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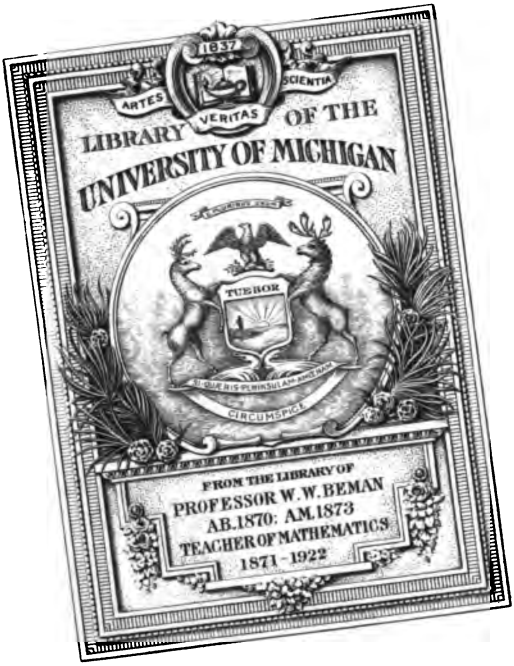

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ADVANCED ALGEBRA

BY

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PREFACE

This text contains essentially the material of the chapters on Algebra in the author's "Advanced Algebra and Trigonometry." Some matter has been added and some, deleted, the form of presentation has been altered in a number of places, many of the exercises have been changed, and all the figures have been redrawn.

The first chapter and portions of the next three chapters are reviews of elementary algebra and are intended to make a close connection with the work of the high school. The preparation of the students in the average freshman class varies so greatly that such reviews are almost indispensable to establish a common basis for further progress.

Graphic methods are introduced early and freely used. This is done both for the immediate utility of these methods and to serve as an introduction to Analytical Geometry.

Numerous simple applications are contained in the exercises and problems, which will serve to establish some connection between theory and practice. Logarithms are introduced in Chapter III, so as to make this method of computation available, early in the course, for the numerical valuation of unknowns from given data.

It is hoped that in the presentation of the theory the requirements of rigor are met satisfactorily so far as that can be done in an elementary course. No effort is made to prove everything, but that which is not proved is clearly pointed out and included in the definitions or assumptions, or excluded from the range of application of the theory.

The later chapters are largely independent of each other and allow a degree of latitude in the selection of subject-matter. In a three-hour course during one semester it is advisable to omit most of the material on simultaneous quadratics to gain time for some of the later subjects.

W. C. BRENKE.

LINCOLN, NEBRASKA
May, 1917.

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ADVANCED ALGEBRA

ADVANCED ALGEBRA

CHAPTER I

Review of the Four Fundamental Operations

1. Letters as Symbols of Quantity. — In algebra, the letters of the alphabet are used to designate quantity or magnitude. Thus we speak of a line whose length is l feet, of a weight of w pounds, or of a velocity of v feet per second. Here the letter used, l, w, v, is suggested by the quantity considered, length, weight, velocity. When a number of different lines are considered, say n lines, their several lengths may be indicated by l_1, l_2, l_3, $\cdots$, l_n, or by $l^{(1)}$, $l^{(2)}$, $l^{(3)}$ $\cdots$, $l^{(n)}$. Three or four different lengths may be indicated by accents (called "primes"), as l', l'', l''', $\cdots$.

Fixed or known quantities are usually designated by the first letters of the alphabet, as by a, b, c, $\cdots$; unknown quantities which are to be determined from given data are represented by the last letters of the alphabet, as by x, y, z, $\cdots$. If x denote a quantity of a certain kind, other quantities of the same kind are indicated by x_1, x_2, x_3, $\cdots$ (read, "x sub-one, x sub-two, x sub-three, etc."), or by $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, $\cdots$ (read "x superscript one, x superscript two, x superscript three, etc."), or by x', x'', x''', $\cdots$ (read "x prime, x second, x third, etc.").

2. Signs of Relation. — These are

$=$, read "equals," "is equal to," etc.;
$\neq$, read "is not equal to";
$<$, read "is less than";
$>$, read "is greater than";
$\nless$, read "is not less than";
$\ngtr$, read "is not greater than";
$\equiv$, read "is identical with";
$\doteq$, read "approaches."

Signs of Aggregation. — When several quantities are to be treated as a single one, they are enclosed by parentheses ( ), brackets [ ], or braces { }, or a line is drawn over them, called a vinculum, ———.

Signs of Quality. — These are

+, positive; −, negative; | |, absolute value.

The first two simply indicate opposite qualities; thus, if $+v$, or simply v, denote a velocity in one direction, then $-v$ denotes an equal velocity in the opposite direction; if $+t$ denote a temperature above zero, $-t$ denotes an equal temperature below zero. The third symbol is used to indicate that we are dealing simply with the numerical (absolute) magnitude of a quantity, without regard to its sign.

3. The Four Fundamental Operations. — These are, addition, subtraction, multiplication and division, indicated by the symbols +, −, ×, ÷, respectively. It will suffice to recall the rules or laws in accordance with which these operations are to be performed. They are here given in the form of equations, and the student is asked to state each in words.

Laws of Addition.

1. If $a = b$ and $c =$ d, then $a + c = b + d$.
2. If $a = b$ and $c \neq d$, then $a + c \neq b + d$.
3. $a + b = b + a$. (*Commutative law.*)
4. $(a + b) + c = a + (b + c)$. (*Associative law.*)

Laws of Subtraction. — (Subtraction defined by $(a - b) + b = a$.)

1. If $a = b$ and $c = d$, then $a - c = b - d$.
2. $(a - c) + b = (a + b) - c$.
3. $a + (b - c) = (a + b) - c$.
4. $(a + c) - b = (a - b) + c$.
5. $(a - c) - b = (a - b) - c$.
6. $a - (b + c) = (a - b) - c$.
7. $a - (b - c) = (a - b) + c$.

Laws of Multiplication.

1. If $a = b$ and $c = d$, then $ac = bd$.
2. If $a = b$ and $c \neq d$, then $ac \neq bd$.
3. $a \times b = b \times a$. (*Commutative law.*)
4. $a \times (b \times c) = (a \times b) \times c$. (*Associative law.*)
5. $(a + b - c) \times d = a \times d + b \times d - c \times d$. (*Distributive law.*)

Laws of Division. — (Division defined by $(a \div b) \times b = a$.)

1. If $a = b$ and $c = d$, then $a \div c = b \div d$, provided $c, d \neq 0$.
2. $(a \div b) \times c = (a \times c) \div b$, provided $b \neq 0$.
3. $a \times (b \div c) = (a \times b) \div c$, provided $c \neq 0$.
4. $(a \div b) \div c = (a \div c) \div b$, provided $b, c \neq 0$.
5. $a \div (b \div c) = (a \div b) \times c$, provided $b, c \neq 0$.

Some Working Rules. — *The sign before a parenthesis may be changed if the sign of each of the terms enclosed is changed also.*

When several quantities are to be subtracted, change their signs and add them.

Division may be expressed as a multiplication of dividend by reciprocal of divisor. (*The reciprocal of a number is 1 divided by that number.*)

The sign of a product will be + or −, according as there are an even or an odd number of negative factors.

4. Rational Numbers. — All positive integers can be formed by adding $+1$ to itself a sufficient number of times. Through the operation of subtraction, negative integers are introduced. By performing the operations of addition, subtraction and multiplication on the system of positive and negative integers, no new numbers are formed. Division, however, does introduce a new class of numbers, namely fractions, positive or negative, formed of the quotient of two integers.

All numbers, positive or negative, which are formed of the quotient of two integers, are called *rational numbers.* They can be obtained from $+1$ by means of the four fundamental operations.

Rational Expressions. — Let there be given certain quantities, $a, b, \cdots x, y, \cdots$. Any expression which can be built up from these quantities by means of the four fundamental operations is called a *rational expression* (or function) in terms of the quantities involved.

5. Zero. — *Zero is defined as that number which may be added to any number without changing the value of the number.* As an equation, the definition is

$$a + 0 = a.$$

Since $$(a - 0) + 0 = a,$$

it also follows that $$a - 0 = a.$$

6. Division by Zero. — The operation of *division by zero is excluded,* because, whatever be the number a, there is no number which represents $a \div 0$. The reason for this we proceed to consider. In the first place, 0 must be less in absolute value than any assignable number, however small. For if this were not the case, we would have $a + 0 \neq a$. Now consider the quotient $\frac{a}{b}$, and suppose a to be fixed, and b to be taken smaller and smaller. As b tends toward zero, the quotient $\frac{a}{b}$ increases without limit and becomes larger than any assignable number. But as b approaches zero, $\frac{a}{b}$ takes the form $\frac{a}{0}$ and at the same time increases without limit so that no value can be assigned to this form.

To illustrate what may happen when we unwittingly divide by zero, consider the following paradox.

Let $x = 1$.
Then $x = x^2$
and $1 - x = 1 - x^2 = (1 + x)(1 - x)$.
Dividing by $1 - x$: $1 = 1 + x$.
Therefore $1 = 2$, since $x = 1$.

We are led to this fallacy by dividing by zero in the form of $1 - x$. Since we assumed $x = 1$, therefore $1 - x = 0$, and hence division by $1 - x$ must be excluded in this problem.

In any expression involving fractions, those cases in which the denominator of any fraction vanishes must be treated as exceptional and especially considered.

If, in a product, a factor approaches zero, while the other factors have any assigned values, then the product approaches zero. This is expressed by the equation $a \times 0 = 0$.

7. Infinity. — A quantity which increases or decreases without limit is said to become infinite. For example, when $b \doteq 0$ ("b approaches zero"), if a is any fixed number, $\frac{a}{b}$ increases without limit. Such quantities, which are larger than any assignable number, are all indicated by the same symbol, ∞ (read "infinity"). As an example, consider the law of gases, *pressure times volume is constant*, that is,

$$pv = c, \quad \text{or} \quad p = \frac{c}{v}.$$

When v is very small (relative to the constant c), p will be very large, and as v becomes still smaller, p must increase. We can choose v so small that p will exceed any assignable quantity, or p becomes ∞ when $v = 0$. This is often indicated by $\lim_{v=0} p = \infty$ (read "the limit of p is infinity, when v approaches zero").

We are thus led to write the equation,

$$\frac{a}{0} = \infty, \text{ when } a \neq 0.$$

This is not a proper equation, but simply an abbreviation for the statement, "*A fraction whose numerator is not zero, and whose denominator approaches zero, becomes larger than any assignable number.*"

Since a quantity which increases without limit can be made as large as we please after being increased or diminished, multiplied or divided by any number, we have

$$\infty + a = \infty, \ \infty - a = \infty; \ \infty \times a = \infty, \ \infty \div a = \infty.$$

8. Powers. — For brevity we put $a \times a = a^2$, $a \times a \times a = a^3$, and $a \times a \times a \cdots$ to n factors $= a^n$. The quantity a^n is called the *nth power* of a. The number n is called the **exponent** and a the **base** of the quantity a^n.

9. Some Important Relations. — The following equations and statements should be verified carefully and committed to memory:

1. $(a + b)^2 = a^2 + 2ab + b^2$.
2. $(a - b)^2 = a^2 - 2ab + b^2$.
3. $a^2 - b^2 = (a + b)(a - b)$.
4. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.
5. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.
6. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc)$.
7. The square of any polynomial equals the sum of the squares of the separate terms plus twice the product of each term by each following term.
8. $a^n - b^n$ is divisible by $(a + b)$ and $(a - b)$ when n is even.
9. $a^n - b^n$ is divisible by $(a - b)$, not by $(a + b)$, when n is odd.
10. $a^n + b^n$ is divisible by $(a + b)$, not by $(a - b)$, when n is odd.
11. $a^n + b^n$ is not divisible by $(a + b)$ or $(a - b)$, when n is even.

10. Exercises. — Simplify, by removal of parentheses and collection of like terms:

1. $(\frac{3}{4}a - \frac{1}{3}) - (\frac{1}{4} - \frac{1}{2}a)$.
2. $(\frac{1}{6}a^2b - \frac{3}{4}ab^2) - (\frac{1}{2}a^2b + \frac{7}{8}ab^2)$.
3. $(0.8a^2 - 3.47ab - 17.25ac) + (\frac{3}{4}a^2 - 0.47ab - 12\frac{5}{8}ac)$.
4. $(\frac{5}{2}x^2 - 3ax - \frac{7}{8}a^2) - (2x^2 - ax - \frac{1}{2}a^2)$.
5. $44x + \{[48y - (6z + 3y - 7x) + 4z] - [48y + 8x - 2z - (4x + y)]\}$.
6. $6a - \{4a - [8b - \overline{2a - b}] + (3b - 4a)\}$.

Perform the operations indicated in the following exercises and simplify the results when possible:

7. $\frac{2}{3}a^2c(\frac{3}{2}b^2 - 6c^2 + \frac{2}{5}ad^2 - 3)$.
8. $3xy^2(x^3 - 3x^2y + 3xy^2 - y^3)$.
9. $0.4ac^2d^4(2a^2b - 3cd^3 + \frac{1}{2}ac^3 - 5)$.
10. $3\frac{1}{2}a^2bc(8a^3 - 4b^2 + 2ab^3 - 3c^2)$.
11. $(x^3 - 2x + 1)(x^3 - 3x - 2)$.
12. $(3a^3b - 2a^2b^2 - ab^3)(2a^2 - ab + 5b^2)$.
13. $(x^4 - 7x^2y^2 + 6xy^3 - y^4)(x^3 - 2xy^2 - y^3)$.
14. $(a + b)^3 - (a - b)^3$.
15. $(\frac{1}{2}a + \frac{1}{3})^3 + (\frac{1}{2}a - \frac{1}{3})^3$.
16. $(x^2 + 1 - y^2 - 3y)(x^2 + 1 + y^2 + 3y)$.
17. $[x^2 + (a + b)x + ab][x^2 - (a + b)x + ab]$.
18. $[(x - a)^2 - ax][(x - a)^2 + ax]$.
19. $a(a + 1)(a + 2) - (a - 1)(a + 2)(a - 3)$.
20. $[x(y - 1) - y(x - 1)][(x + y)^2 - (x - y)^2]$.
21. $\frac{15}{8}m^4np^5 \div -\frac{5}{4}m^2np^4$.
22. $a^2bc^7 \div \frac{5}{11}a^4b^2c^8$.
23. $\frac{15}{16}x^4y^5 \div -\frac{81}{32}x^3y^3$.
24. $3a^2(b + x)^3 \div 6a^3(b + x)^2$.
25. $1.75x^3(x^2 - 1)^4 \div 25x^5(1 - x^2)^2$.
26. $(8a^5b - 24a^4b^3 + 16a^7b^8) \div -8a^3b$.
27. $(8x^3y - \frac{5}{6}xy^4 - \frac{1}{3}y^5 + 2y^2) \div -\frac{4}{5}x^3y^2$.
28. $x^5(a^2 + b^2) - 2x^4(a^2 + b^2)^3 \div x^3(a^2 + b^2)^2$.
29. $(6a^3x - 17a^2x^2 + 14ax^3 - 3x^4) \div (2a - 3x)$.
30. $(4y^4 - 18y^3 + 22y^2 - 7y + 5) \div (2y - 5)$.
31. $[2x^3 + 21x^2y - 81y^2(x + y)] \div (2x - 9y)$.
32. $(-\frac{1}{8}d^5 + \frac{5}{6}d^4 - \frac{41}{24}d^3 + d^2) \div (-\frac{3}{4}d^2 + 2d)$.
33. $(\frac{9}{16}a^4 + \frac{7}{8}a^3b + \frac{19}{36}a^2b^2 - \frac{1}{6}ab^3) \div (\frac{3}{2}a - \frac{1}{3}b)$.
34. $(\frac{1}{14}m^3 - \frac{1}{24}m^2n - \frac{25}{12}mn^2 - \frac{14}{9}n^3) \div (\frac{1}{2}m + \frac{7}{3}n)$.
35. $(x^5 - \frac{23}{30}x^4 + \frac{81}{10}x^3 - \frac{7}{3}x^2 - \frac{181}{18}x + \frac{5}{3}) \div (x^2 - \frac{1}{6}x + 5)$.
36. $(2a^3 - 16a + 6) \div (a + 3)$.
37. $(4x^4 - x^2y^2 - 6xy^3 - 9y^4) \div (2x^2 + xy + 3y^2)$.
38. $(x^4 + 4x^2y^2 - 32y^4) \div (x - 2y)$.
39. $(a^5 - 5a^3b^2 + 5a^2b^3 - b^5) \div (a^2 + 3ab + b^2)$.
40. $(x^3 + 8y^3) \div (x + 2y)$.
41. $(\frac{1}{81}x^4 - 9y^2) \div (\frac{1}{9}x^2 - 3y)$.

42. $(27\,a^3b^3 - 64\,x^3y^3) \div (3\,ab - 4\,xy)$.
43. $(a^3b^3 - c^3) \div (a^2b^2 + abc + c^2)$.
44. $(u^5 + 32\,v^5) \div (u + 2\,v)$.
45. $(a - b + c - d)^2$.
46. $(x - \frac{1}{2}\,y - 2\,u - w)^2$.

11. Factoring. — To *factor an expression* is to find two or more quantities whose product equals the given expression. When two or more expressions contain the same factor, it is called their *common factor*.

We shall illustrate the methods commonly used in factoring given expressions by means of some typical examples.

(*a*) *Expressions, all of whose terms contain a common factor.*

Example. $\frac{1}{6}\,x^3y^3z^3 + \frac{1}{9}\,x^2y^2z - \frac{1}{21}\,x^4y^3z^2 = \frac{1}{3}\,x^2y^2z\,(\frac{1}{2}\,xz^2 + \frac{1}{3} - \frac{1}{7}\,x^2yz)$.

(*b*) *Expressions whose terms can be grouped, so that each group contains the same factor.*

Example.
$$\begin{aligned} x^3 - 7\,x^2y + 14\,xy^2 - 8\,y^3 &= (x^3 - 8\,y^3) - (7\,x^2y - 14\,xy^2) \\ &= (x - 2\,y)\,(x^2 + 2\,xy + 4\,y^2) - 7\,xy\,(x - 2\,y) \\ &= (x - 2\,y)\,(x^2 - 5\,xy + 4\,y^2) \\ &= (x - 2\,y)\,(x - y)\,(x - 4\,y). \end{aligned}$$

(*c*) *Trinomials of the form* $ax^2 + bx + c$.

Let h, k be a pair of factors whose product is a, and m, n a pair whose product is c. Arrange these four factors as in the adjacent schemes $\begin{smallmatrix} h & & n \\ & \times & \\ k & & m \end{smallmatrix}$ $\begin{smallmatrix} h & & m \\ & \times & \\ k & & n \end{smallmatrix}$ and form the cross-products as indicated. The sum of the cross-products must equal b. If this is true in the first scheme, the factors are $(hx + n)\,(kx + m)$; in the second, the factors are $(hx + m)\,(kx + n)$.

Example. $12\,x^2 - 7\,x - 10$.

Here h, k may be one of the pairs of numbers 1, 12, or 2, 6, or 3, 4, both numbers to be taken with the same sign. The numbers m, n may be -1, 10, or $+1$, -10, or -2, 5, or $+2$, -5. By trial we find that h, k must be 3, 4, and m, n must be 2, -5. The factors are therefore $(3\,x + 2)\,(4\,x - 5)$.

To find the factors of $12\,x^2 - 7\,xy - 10\,y^2$, we would proceed as above and obtain $(3\,x + 2\,y)\,(4\,x - 5\,y)$.

(*d*) *Expressions which can be written as the difference of the squares of two quantities.*

The factors are the sum and the difference of the two quantities respectively.

Example.
$$\begin{aligned} a^4 + a^2b^2 + b^4 &= a^4 + 2\,a^2b^2 + b^4 - a^2b^2 \\ &= (a^2 + b^2)^2 - (ab)^2 \\ &= (a^2 + ab + b^2)\,(a^2 - ab + b^2). \end{aligned}$$

(*e*) *Expressions of the form $P^2 + 2\,PQ + Q^2$, where P and Q are monomials or polynomials.*

The expression is then the product of two factors each equal to $(P + Q)$, and is therefore $(P + Q)^2$.

Example.
$$\begin{aligned} & x^2 + y^2 - 2\,xy - 4\,ax + 4\,ay + 4\,a^2 \\ &= (x - y)^2 - 4\,a\,(x - y) + 4\,a^2 \\ &= (x - y - 2\,a)^2. \end{aligned}$$

(*f*) **Factor Theorem.** — If a polynomial in x reduces to zero when x is replaced by h, the polynomial contains the factor $(x - h)$.

Proof: Let the polynomial be

$$P \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n.$$

Putting h for x, we have by hypothesis

$$a_0h^n + a_1h^{n-1} + a_2h^{n-2} + \cdots + a_{n-1}h + a_n = 0.$$

Therefore by subtraction,

$$P = a_0(x^n - h^n) + a_1\,(x^{n-1} - h^{n-1}) + a_2\,(x^{n-2} - h^{n-2}) + \cdots + a_{n-1}\,(x - h).$$

But each term of the right member of the last equation contains the factor $x - h$. (See 8 and 9 of **9**.) Hence P is divisible by $(x - h)$.

Example. Factor $x^3 + 3\,x^2 - 4\,x - 12$.

If this is the product of three factors $(x - h)\,(x - k)\,(x - l)$, then evidently $hkl = 12$. Hence we substitute in the given polynomial the factors of 12, and find that it vanishes when $x = 2$, $x = -2$, and $x = -3$. Hence the factors are $(x - 2)\,(x + 2)\,(x + 3)$.

12. Exercises. — Factor:

1. $\dfrac{x^2}{2\,a} - \dfrac{xy}{8\,a^3} - \dfrac{x}{4\,a^2}$.
2. $x^2 + 2\,ax + a^2 - y^2$.
3. $3\,x^2 + 4\,x + 1$.
4. $15\,x^2 + 7\,x - 2$.
5. $6\,x^2 - 19\,xy - 7\,y^2$.
6. $x^2 + 2\,x - 24$.
7. $8\,x^4 + 27\,xy^3$.
8. $27\,x^4 - 8\,xy^3$.
9. $x^4 - 13\,x^2 + 36$.
10. $9\,a^4 - 13\,a^2 + 4$.
11. $x^2 + x - 110$.
12. $7 - 10\,x + 3\,x^2$.
13. $x^2 + 10\,a^2x + 9\,a^4$.
14. $(a + b)^4 - 8$.
15. $x^2y^2 + 3\,xyz - 10\,z^2$.
16. $2 + 7\,x - 15\,x^2$.
17. $x^3 - 36\,x - x^2 + 36$.
18. $a^8 - 1$.
19. $x^6 + 1$.
20. $(a + b)^3 - 1$.
21. $(x^3 + y^3) - (x + y)^3$.

22. $a^4 + b^4 - c^4 - d^4 - 2a^2b^2 + 2c^2d^2$.

23. $a^2c^2 + acd - abc - bd$.

24. $1 - a^2x^2 - b^2y^2 + 2abxy$.

25. $x^4y - x^2y^3 + x^3y^2 - xy^4$.

26. $a^8 - 82a^4 + 81$.

27. $x^4y^2 - 17x^2y - 110$.

28. $(a^2 + 3)^2 - 36a^2$.

29. $x^3 + 9x^2 + 16x + 4$.

30. $x^4 + 7x^3 + x^2 - 63x - 90$. Ans. $= (x + 2)(x + 3)(x + 5)(x - 3)$.

13. Highest Common Factor. — The *highest common factor* (*H. C. F.*) of two or more polynomials is the polynomial of highest degree that will divide them all without a remainder.

When each of the given polynomials can be factored by inspection, the *H. C. F.* is easily determined from their common factors.

Example. The *H. C. F.* of $32(x - 1)^2(x + 1)^3(x^2 + 1)$ and $24(x - 1)^3(x + 1)^2(x^2 + 1)^2$ is $8(x - 1)^2(x + 1)^2(x^2 + 1)$.

When the given polynomials cannot be readily factored, we use a method like that of arithmetic.

Let the given polynomials be P_1 and P_2 and let Q be the quotient and R the remainder when P_1 is divided by P_2. Then

$$P_1 = P_2Q + R.$$

Hence any factor common to P_1 and P_2 is also a factor of R. Hence it is a common factor of P_2 and R. Divide P_2 by R, obtaining

$$P_2 = RQ_1 + R_1.$$

Hence a common factor of P_2 and R is also a factor of R_1. Dividing R by R_1, we obtain

$$R = R_1Q_2 + R_2,$$

and the common factor must be present in R_2, and so on.

Rule. — If at any step there is no remainder, the last divisor is the required *H. C. F.*

14. Least Common Multiple. — The *least common multiple* (*L. C. M.*) of two or more polynomials is the polynomial of lowest degree that is exactly divisible by each of them.

When the given polynomials can be easily factored by inspection, form the product of all the types of factors present in any of them, taking each factor the greatest number of times that it occurs in any of the given expressions; this product is their *L. C. M.*

When the given polynomials cannot readily be factored, their *L. C. M.* is obtained by use of the following theorem:

The product of the H. C. F. and L. C. M. of two polynomials equals the product of the polynomials.

Proof: Let F be the $H. C. F.$, and M the $L. C. M.$ of the two polynomials P_1 and P_2. Also let

$$\frac{P_1}{F} = Q_1 \text{ and } \frac{P_2}{F} = Q_2;$$

then $$P_1 = FQ_1 \text{ and } P_2 = FQ_2.$$

Hence $$P_1P_2 = F \times FQ_1Q_2.$$

Since F contains all factors common to P_1 and P_2, Q_1 and Q_2 have no common factor, and the product FQ_1Q_2 contains all the factors of the types present in both P_1 and P_2.

$$\therefore \quad M = FQ_1Q_2 = \frac{P_1P_2}{F}; \text{ or, } MF = P_1P_2.$$

Rule. — To find the $L. C. M.$ of two polynomials, divide their product by their $H. C. F.$

To find the $L. C. M.$ of more than two polynomials, find the $L. C. M.$ of two of them, then the $L. C. M.$ of this and a third one of the polynomials, and so on.

15. Exercises. — Find the $H. C. F.$ of

1. $6(x+1)^3$ and $9(x^2-1)$.

2. a^6-b^6 and a^4-b^4.

3. $12(x^2+y^2)^2$ and $8(x^4-y^4)$.

4. u^5-v^5 and u^2-v^2.

5. $(a^2x-ax^2)^2$ and $ax(a^2-x^2)^2$.

6. $27(a^4-b^4)$ and $18(a+b)^2$.

7. $(24a^2+36ab-48ac)$ and $(30a^3+45a^2b-60a^2c)$.

8. $125x^3-1$ and $35x^2-7x+5ax-a$.

9. $4x^2-12xy+9y^2$ and $4x^2-9y^2$.

10. $x^2+2x-120$ and $x^2-2x-80$.

11. $12x^2-17ax+6a^2$ and $9x^2+6ax-8a^2$.

12. x^3+4x^2-5x and x^2-6x+5.

13. $x^3+3x^2+7x+21$ and $2x^4+19x^2+35$.

14. $a^4+7a^3+7a^2-15a$ and $a^3-2a^2-13a+110$.

15. $20x^4+x^2-1$ and $75x^4+15x^3-3x-3$.

16. $x^4-ax^3-a^2x^2-a^3x-2a^4$ and $3x^3-7ax^2+3a^2x-7a^3$.

17. x^4-y^4, x^3+y^3, and x^5+y^5.

18. x^2-2a^2-ax, x^2-6a^2+ax, and x^2-8a^2+2ax.

19. $a^4+a^2b^2+b^4$, a^4+ab^3, and a^3b+b^4.

20. $3x^3-7x^2y+5xy^2-y^3$, $x^2y+3xy^2-3x^3-y^3$, and $3x^3+5x^2y+xy^2-y^3$.

Find the $L. C. M.$ of:

21. $8a^2x^3y^3$ and $12abx^2y^3$.

22. $4\,ax^3y^2$, $6\,a^2xy^3$, and $18\,a^3x^2y$.
23. $a^2 - b^2$ and $(a - b)^2$.
24. $a^2bx - ab^2y$ and $abx + b^2y$.
25. $x^2 - 3\,x - 4$ and $x^2 - x - 12$.
26. $x^2 - 1$ and $x^2 + 4\,x + 3$.
27. $6\,x^2 + 5\,x - 6$ and $6\,x^2 - 13\,x + 6$.
28. $12\,x^2 + 5\,x - 3$ and $6\,x^3 + x^2 - x$.
29. $12\,x^2 - 17\,ax + 6\,a^2$ and $9\,x^2 + 6\,ax - 8\,a^2$.
30. $a^3 - 9\,a^2 + 23\,a - 15$ and $a^2 - 8\,a + 7$.
31. $m^3 + 2\,m^2n - mn^2 - 2\,n^3$ and $m^3 - 2\,m^2n - mn^2 + 2\,n^3$.
32. $x^2 - y^2$, $(x - y)^2$, and $x + y$.
33. $x^2 + 3\,x + 2$, $x^2 + 4\,x + 3$, and $x^2 + 5\,x + 6$.
34. $x^2 + 5\,x + 10$, $x^3 - 19\,x - 30$, and $x^3 - 15\,x - 50$.
35. $x^2 + 2\,x - 3$, $x^3 + 3\,x^2 - x - 3$, and $x^3 + 4\,x^2 + x - 6$.
36. $6\,x^2 - 13\,x + 6$, $6\,x^2 + 5\,x - 6$, and $9\,x^2 - 4$.
37. $x^3 - 1$, $x^2 + 1$, and $x^3 + 1$.
38. $x^2 + 1$, $x^4 - 1$, and $x^6 - 1$.
39. $a^3 - b^3$, $a^9 - b^9$, and $a^6 - b^6$.
40. $x^2 - y^2$, $x^3 + y^3$, $x^3 - y^3$, and $x^6 + y^6$.

16. Fractions. — An algebraic fraction is the indicated quotient of two algebraic expressions. It is written in the form $\frac{N}{D}$, N being called the numerator and D the denominator.

When N and D have a common factor F, so that we may put

$$N = N_1F \text{ and } D = D_1F,$$

then the fraction may be simplified as follows:

$$\frac{N}{D} = \frac{N_1F}{D_1F} = \frac{N_1}{D_1}.$$

When all factors common to N and D have been removed in this way, the fraction is said to be *reduced to its lowest terms.*

When the common factors of N and D are not obvious on inspection, find the $H.\ C.\ F.$ of N and D, and remove it as above.

17. Sign of a Fraction. — By the rules for division we have,

$$\frac{N}{D} = -\frac{-N}{D} = -\frac{N}{-D} = \frac{-N}{-D}.$$

Hence the rules: *Changing the sign of either numerator or denominator changes the sign of the fraction.*

Changing the signs of both numerator and denominator does not affect the sign of the fraction.

The sign of a fraction may be changed either by changing the sign standing before the fraction, or by changing the sign of the numerator or of the denominator.

18. Reduction of Fractions. — An **integral expression** is one whose literal parts are free from fractions.

A **mixed expression** is one formed from the sum of an integral part and one or more fractions.

A **complex fraction** is one whose numerator, or denominator, or both are fractions or mixed expressions.

Every mixed expression and every complex fraction can be reduced to a simple fraction (*or to an integral expression*).

For, two or more simple fractions can be reduced to a *common denominator* and then combined into a single fraction by writing the sum of the numerators over the common denominator. For this purpose the simplest common denominator is the *L. C. M.* of the separate denominators. This is called the **least common denominator** of the fractions considered. In this manner we reduce

$$\frac{N_1}{D_1}+\frac{N_2}{D_2}+\cdots \text{ to } \frac{N}{D}.$$

A mixed expression is reduced by the formula

$$P+\frac{N}{D}=\frac{PD+N}{D}.$$

Finally, a complex fraction is reduced by first reducing its numerator and denominator separately to simple fractions. The reduction is then completed by the formula,

$$\frac{\frac{N}{D}}{\frac{N'}{D'}}=\frac{N}{D}\times\frac{D'}{N'}=\frac{ND'}{N'D}.$$

Examples.

1. Simplify $$\frac{2}{\frac{1}{x}-\frac{1}{y}}+\frac{y}{1-\frac{y}{x}}-\frac{x}{1-\frac{x}{y}}.$$

First reduce each fraction to a simple fraction, thus:

$$\frac{2}{\frac{1}{x}-\frac{1}{y}}=\frac{2}{\frac{y-x}{xy}}=\frac{2xy}{y-x},$$

$$\frac{y}{1-\frac{y}{x}}=\frac{y}{\frac{x-y}{x}}=\frac{xy}{x-y},$$

$$-\frac{x}{1-\frac{x}{y}}=-\frac{x}{\frac{y-x}{y}}=-\frac{xy}{y-x}=\frac{xy}{x-y}.$$

Reducing to the common denominator $x - y$, we have

$$\frac{2xy}{y-x}+\frac{xy}{x-y}+\frac{xy}{x-y}=\frac{-2xy+xy+xy}{x-y}=0 \text{ (provided } x \neq y\text{).}$$

2. $$x^3-\frac{x^2}{x+\dfrac{1-x^4}{x-\dfrac{1}{x}}}=x^3-\frac{x^2}{x+\dfrac{1-x^4}{\dfrac{x^2-1}{x}}}=x^3-\frac{x^2}{x+\dfrac{x(1-x^4)}{x^2-1}}$$

$$=x^3-\frac{x^2}{x-x(1+x^2)}=x^3-\frac{x^2}{-x^3}=x^3+\frac{1}{x}=\frac{x^4+1}{x}.$$

19. Exercises. — Reduce to simple fractions or to integral expressions:

1. $\left(\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}\right)(a^2-x^2).$
2. $\dfrac{a^3-b^3}{a^3+b^3}\div\dfrac{a-b}{a+b}.$
3. $\dfrac{a^4-b^4}{x^3+y^3}\times\dfrac{x^2-xy+y^2}{a^2+b^2}.$
4. $\left(\dfrac{x}{a}+\dfrac{y}{b}\right)\left(\dfrac{a}{x}-\dfrac{b}{y}\right).$
5. $\left(\dfrac{x^5}{y^5}+\dfrac{y^5}{x^5}\right)\div\left(\dfrac{x}{y}+\dfrac{y}{x}\right).$
6. $\dfrac{x^{12}+y^{12}}{x^{12}-y^{12}}\div\dfrac{x^4+y^4}{x^4-y^4}.$
7. $\dfrac{x^2+7x+12}{x^2-x-12}\div\dfrac{x^2+6x+8}{x^2-2x-8}.$
8. $\dfrac{\dfrac{x+1}{x}-\dfrac{y+1}{y}}{\dfrac{1}{x}-\dfrac{1}{y}}.$
9. $1-\dfrac{1}{x+\dfrac{1}{x}}.$
10. $\dfrac{\dfrac{x}{x+y}+\dfrac{y}{x-y}}{\dfrac{x}{x-y}-\dfrac{y}{x+y}}.$
11. $\dfrac{x^6+y^6}{x^2-y^2}\div\dfrac{3x^2+3y^2}{x+y}.$
12. $\dfrac{45(x-y)}{32(z+y)}\div\dfrac{27(x-y)^3}{128b(z+y)^3}.$
13. $\dfrac{a^2-4x^2}{a^2+4ax}\div\dfrac{a^2+2ax}{ax+4x^2}.$
14. $\dfrac{a^2+ab}{a^2+b^2}\div\dfrac{ab(a+b)^2}{a^4-b^4}.$
15. $\dfrac{u^3+v^3}{u^2v^2-u^4}\div\dfrac{u^2-uv+v^2}{uv^2-v^3}.$
16. $\dfrac{p^2+3p+9}{p^4-3p^2+9}\div\dfrac{p^3-27}{p^6+27}.$
17. $\dfrac{x^6-y^6}{(x+y)^2}\div\dfrac{x^2-xy+y^2}{x+y}.$
18. $\dfrac{\dfrac{x-y}{x+y}-\dfrac{x^3-y^3}{x^3+y^3}}{\dfrac{x+y}{x-y}-\dfrac{x^2+y^2}{x^2-y^2}}.$
19. $\dfrac{1}{y+\dfrac{1}{1+\dfrac{1+y}{3-y}}}.$
20. $\dfrac{1}{x(x-1)}+\dfrac{2}{1-x^2}+\dfrac{1}{x(x+1)}.$
21. $\dfrac{1}{x^2-5x+6}-\dfrac{2}{x^2-4x+3}+\dfrac{1}{x^2-3x+2}.$
22. $\dfrac{u}{u+v}+\dfrac{u}{u-v}+\dfrac{2u^2}{u^2+v^2}.$
23. $\dfrac{x-1}{x^2-5x+6}-\dfrac{2(x-2)}{x^2-4x+3}+\dfrac{x-3}{x^2-3x+2}.$

24. $\frac{7+3x^2}{4-x^2}-\frac{5-2x^2}{4+4x+x^2}-\frac{3-2x+x^2}{4-4x+x^2}.$

25. $\frac{1-2x}{3(x^2-x+1)}-\frac{2x-3}{2(x^2+1)}+\frac{1}{6(x+1)}.$

26. $\left(\frac{a}{bc}-\frac{b}{ac}-\frac{c}{ab}-\frac{2}{a}\right)\left(1-\frac{2c}{a+b+c}\right).$

27. $\left(\frac{x^2}{a^2}+\frac{a^2}{x^2}-\frac{x}{a}-\frac{a}{x}+1\right)\left(\frac{x}{a}-\frac{a}{x}\right).$

28. $\left(\frac{x}{a}-\frac{a}{x}+\frac{y}{b}-\frac{b}{y}\right)\left(\frac{x}{a}-\frac{a}{x}-\frac{y}{b}+\frac{b}{y}\right).$

29. $\left(1-\frac{7x}{11y}\right)\left(\frac{7x}{11y}+\frac{49x^2}{121y^2}+\frac{343x^3}{1331y^3}\right).$

30. $\left(1-\frac{2x-3y}{2x+3y}\right)\left(2+\frac{6y}{2x-3y}\right).$

31. $\left(\frac{ab}{3c^2}-\frac{3bc}{5a^2}\right)\left(\frac{5ac}{7b^2}-\frac{7ab}{9c^2}\right)\left(\frac{3b}{5a^2}-\frac{ab}{3c^2}\right).$

32. $\left(\frac{4x^3y^2}{5a^3}-\frac{3x^2y^3}{2a^2b}+\frac{2xy^4}{3ab^2}-\frac{y^5}{b^3}\right)\left(\frac{2x^2y}{3a^2}-\frac{3xy^2}{5ab}-\frac{3y^3}{2b^2}\right).$

CHAPTER II

THEORY OF EXPONENTS. SURDS. IMAGINARIES.

20. Involution. — The operation of raising a quantity to an indicated power is called *involution.*

In (8) the symbol a^n was defined to mean

$$a \times a \times a \times \cdots \text{ to } n \text{ factors.}$$

21. To Multiply a^n by a^m. — By direct multiplication, we have

$$a^2 \times a^3 = a^5 = a^{2+3};$$
$$a^3 \times a^5 = a^8 = a^{3+5};$$

and in general, if n and m are two positive integers,

$$\text{I.} \qquad a^n \times a^m = a^{n+m}.$$

Rule. To multiply a^n by a^m, add the exponents; the product is a^{n+m}.

22. To divide a^n by a^m. — By cancellation we see that

$$a^5 \div a^2 = a^3 = a^{5-3};$$
$$a^9 \div a^4 = a^5 = a^{9-4};$$

and in general, when n and m are positive integers and n is greater than m,

$$\text{II.} \qquad a^n \div a^m = a^{n-m}.$$

Now suppose that n is less than m; thus,

$$\frac{a^2}{a^5} = \frac{1}{a^3}.$$

If we apply formula II to this case we would have

$$a^2 \div a^5 = a^{2-5} = a^{-3}.$$

To make this agree with the result obtained by direct cancellation we must regard a^{-3} as meaning $\frac{1}{a^3}$. That is a^{-3} is simply another way of writing $\frac{1}{a \times a \times a}$.

Negative Exponent. — We shall now define the symbol a^{-n} to mean $\frac{1}{a^n}$, n being a positive integer.

III. $$a^{-n} \equiv \frac{1}{a^n}.$$

Then formula II can be used even when n is less than m. Thus:

$$a^4 \div a^9 = a^{4-9} = a^{-5} = \frac{1}{a^5}.$$

Rule. To divide a^n by a^m, subtract m from n; the quotient is a^{n-m}.

Zero Exponent. — Suppose $n = m$. Then II gives

$$a^n \div a^n = a^{n-n} = a^0.$$

But $$a^n \div a^n = 1.$$

Hence a^0 must be regarded as equal to 1.

IV. $$a^0 = 1.$$

23. To Raise a^n to the mth Power. — As examples,

$$(a^2)^3 = a^2 \cdot a^2 \cdot a^2 = a^6 = a^{2\times 3};$$
$$(a^5)^4 = a^5 \cdot a^5 \cdot a^5 \cdot a^5 = a^{20} = a^{5\times 4}.$$

In general, when n and m are positive integers,

V. $$(a^n)^m = a^{nm}.$$

Rule. To raise a^n to the power m, multiply n by m; the result is $a^{n\times m}$.

24. To Multiply or Divide a^n by b^n. — By trial we see that

$$a^2 \cdot b^2 = a \times a \cdot b \times b = a \times b \cdot a \times b = (ab)^2;$$
$$a^3 \cdot b^3 = ab \cdot ab \cdot ab = (ab)^3;$$

and in general, if n is an integer,

VI. $$a^n \cdot b^n = (ab)^n.$$

Similarly for division:

VII. $$a^n \div b^n = (a \div b)^n.$$

Hence we have the following formulas for dealing with integral exponents.

I. $a^n \cdot a^m = a^{n+m}.$

II. $\dfrac{a^n}{a^m} = a^{n-m}.$

III. $a^{-n} = \dfrac{1}{a^n}.$

IV. $a^0 = 1.$

V. $(a^n)^m = a^{nm}.$

VI. $a^n b^n = (ab)^n.$

VII. $\dfrac{a^n}{b^n} = \left(\dfrac{a}{b}\right)^n.$

25. Exercises. — Reduce to simplest form:

1. $2^2 \times 3^3$.
2. $3^2 \times 4^2$.
3. $3^2 \times 2^3 \times 4^2$.
4. $a^6 \cdot a^5$.
5. $a^q \cdot a^p$.
6. $a^p \cdot a^q \cdot a^r$.
7. $12^4 \div 4^4$.
8. $6^5 \div 3^5$.
9. $(2a)^8 \div (2a)^5$.
10. $\dfrac{a^6 \cdot a^5}{a^8 \cdot a^2}$.
11. $\dfrac{b^4 \cdot b^{-7}}{b^2}$.
12. $\dfrac{3^{-3}x^5}{3^{-2}x^{-3}}$
13. $2a^3b^2c^4 \cdot 3a^2b^5c^5$.
14. $5a^{-4}b^2c^{-3} \cdot 4a^3b^{-5}c^{-2}$.
15. $6p^5q^6r^{-3} \div 2p^{-3}q^4r^{-2}$.
16. $(x^{-1} + y^{-1})^2$.
17. $(x^{-2} + y^{-2})(x^{-2} - y^{-2})$.
18. $(x^0 + y^0)(x^0 - y^0)$.
19. $[(2a^2)^3]^2$.
20. $[(3x^{-1})^2]^{-3}$.

26. Fractional Exponents. Roots. — If the area of a square is 3 square units, what is the side of the square? The answer must be — *that number whose square is 3.* It is called the *square root* of 3, and is indicated by one of the symbols

$$\sqrt{3} \quad \text{or} \quad 3^{\frac{1}{2}}.$$

Likewise, if the volume of a cube is 5 cubic units, its edge must be a *number whose cube is 5.* It is called the *cube root* of 5, and its symbol is

$$\sqrt[3]{5} \quad \text{or} \quad 5^{\frac{1}{3}}.$$

What is the area of one face of this cube? It is a square whose side is $5^{\frac{1}{3}}$, hence its area is $5^{\frac{1}{3}} \times 5^{\frac{1}{3}} = (5^{\frac{1}{3}})^2$.

Similarly, if the volume of a cube is v cubic units,

one edge $= \sqrt[3]{v}$ or $v^{\frac{1}{3}}$ linear units;
one face $= v^{\frac{1}{3}} \times v^{\frac{1}{3}}$ or $(v^{\frac{1}{3}})^2$ square units.

The expression $(v^{\frac{1}{3}})^2$ we usually write in the form $v^{\frac{2}{3}}$, using Rule V as though the exponents were integers. Likewise we transfer all our other rules to fractional exponents, and find that they give correct results when we give proper meanings to the fractional exponents. These meanings we shall now consider.

Definition. The symbol $\sqrt[n]{a}$ *or* $a^{\frac{1}{n}}$, *called the nth root of a, shall be understood to mean that number whose nth power is a.*

That is,

$$(a) \qquad \left(a^{\frac{1}{n}}\right)^n = a.$$

This agrees with Rule V, since

$$\left(a^{\frac{1}{n}}\right)^n = a^{\frac{1}{n}\times n} = a^1 = a.$$

By applying Rule V to either of the expressions $(a^{\frac{1}{3}})^2$ or $(a^2)^{\frac{1}{3}}$ we would get $a^{\frac{2}{3}}$. Hence we may regard $a^{\frac{2}{3}}$ to mean the square of the cube root of a, or, the cube root of the square of a.

Likewise, by Rule V, the symbol $a^{\frac{m}{n}}$ may be obtained from either of the expressions

$$\left(a^{\frac{1}{n}}\right)^m \quad \text{or} \quad (a^m)^{\frac{1}{n}};$$

Hence the expression $a^{\frac{m}{n}}$ *may be regarded*

(b) (1) *as the mth power of the nth root of a,*
(2) *or, as the nth root of the mth power of a.*

We shall further assume that

$$(c) \qquad a^{-\frac{m}{n}} \textit{ shall mean } 1 \div a^{\frac{m}{n}}$$

as in Rule III for integral exponents.

With these meanings, (a), (b), (c), attached to the fractional exponents, it is easy to show that all the rules for integral exponents apply also to fractional exponents.

In the first place, we can show that

$$\left(a^{\frac{p}{q}}\right)^m = a^{\frac{pm}{q}}.$$

For $$\left(a^{\frac{p}{q}}\right)^m = \left[\left(a^{\frac{1}{q}}\right)^p\right]^m = \left(a^{\frac{1}{q}}\right)^{pm} = a^{\frac{pm}{q}}.$$

Using this result we can verify that

$$(1) \qquad a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{p}{q}+\frac{r}{s}};$$

$$(2) \qquad a^{\frac{p}{q}} \div a^{\frac{r}{s}} = a^{\frac{p}{q}-\frac{r}{s}};$$

$$(3) \qquad \left(a^{\frac{p}{q}}\right)^{\frac{r}{s}} = a^{\frac{pr}{qs}}.$$

For, raising both sides of each equation to the power qs and simplifying results we have identically true equations, namely

$$(1') \qquad a^{ps+qr} = a^{ps+qr};$$

$$(2') \qquad a^{ps-qr} = a^{ps-qr};$$

$$(3') \qquad a^{pr} = a^{pr}.$$

27. Principal Root. — It can be shown that any number has q distinct qth roots. That is, the symbol $a^{\frac{p}{q}}$ or $\sqrt[q]{a^p}$ may have q distinct values. One of these values is selected as the *principal root.* It is the ordinary arithmetic root when one exists; thus $\sqrt[4]{81} = 3$; $\sqrt[3]{-64} = -4$; and so on. It may be that all the roots are imaginary (**34**); then any one of them may be called the principal root.

In this text the radical sign $\sqrt[q]{}$ will be used to mean the principal root only. Thus, $\sqrt{4} = 2$, not ± 2. Such restriction is convenient if we wish to argue as follows:

if $\qquad a = b, \quad$ then $\quad \sqrt[q]{a} = \sqrt[q]{b}.$

This is the type of reasoning used to prove equations (1), (2), (3) of (**26**) from equations (1′), (2′), (3′).

28. A Paradox. Students commonly overlook the fact that any number has *two* square roots, *three* cube roots, and so on. See (**27**). If $a^2 = b^2$, does $a = b$? Such a conclusion may be quite wrong, as illustrated below.

Let $\qquad W =$ the weight of an elephant;
and $\qquad w =$ the weight of a gnat.
To prove $\qquad W = w.$

Proof: Let their combined weights be $2x$; that is,
(1) $\qquad W + w = 2x.$
(2) Hence $\qquad W = 2x - w,$
(3) and $\qquad W - 2x = -w.$
Multiply (2) by (3):
(4) $\qquad W^2 - 2Wx = w^2 - 2wx.$
Add x^2 to both members of (4):
(5) $\qquad W^2 - 2Wx + x^2 = w^2 - 2wx + x^2.$
Taking square roots:
(6) $\qquad W - x = w - x.$
Therefore $\qquad W = w.$

Equation (5) is a case of $a^2 = b^2$. From this there are *two* possible conclusions, either $a = b$ or $a = -b$. The first conclusion is wrong in the above illustration, because it contradicts (1). The second conclusion is correct, and does not lead to the ridiculous result that $W = w$. It simply leads back to (1).

29. Exercises.

Simplify the following, performing the indicated operations where possible.

1. $8^{\frac{2}{3}} \cdot 16^{\frac{1}{4}}.$ **2.** $81^{\frac{3}{4}} \cdot 27^{\frac{2}{3}}.$
3. $12\,a^{\frac{2}{3}} \div 3\,a^{\frac{1}{3}}.$ **4.** $(8\,a^{\frac{2}{3}}b^{\frac{1}{2}}) \div (4\,a^{\frac{1}{3}}b^{\frac{1}{4}}).$
5. $(\sqrt{2}\,x^{\frac{3}{5}}\,x^{-\frac{2}{5}})^2.$ **6.** $\sqrt{x}\,(x^{\frac{3}{2}})^{\frac{2}{3}}.$
7. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})\,(a^{\frac{1}{2}} - b^{\frac{1}{2}}).$ **8.** $(x^{\frac{1}{2}} + y^{\frac{1}{2}})^2.$

9. $(x - y) \div (x^{\frac{1}{3}} - y^{\frac{1}{3}})$. **10.** $(x + y) \div (x^{\frac{1}{3}} + y^{\frac{1}{3}})$.

11. $(a^{\frac{2}{3}} - b^{\frac{2}{3}}) \div (a^{\frac{1}{3}} + b^{\frac{1}{3}})$. **12.** $\sqrt{a^{\frac{2}{3}} - 2a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}}$.

Express the following with fractional exponents, in simplest form.

13. $\sqrt{x^5} \cdot \sqrt[3]{x^2}$. **14.** $\sqrt{x\sqrt{x}}$.

15. $\sqrt[3]{a^2\sqrt[3]{a}}$. **16.** $\sqrt[5]{5\sqrt[5]{5}}$.

17. $\sqrt[4]{\sqrt[3]{a^2} \cdot \sqrt{b^3}}$. **18.** $\sqrt[4]{c^3} \div \sqrt[3]{c^4}$.

30. Irrational Numbers. — By the use of fractional exponents we are led to numbers which cannot be produced from integers by means of the four fundamental operations. Thus, if we attempt to calculate $\sqrt{2}$ we are led to a non-terminating decimal. To four decimals we have

$$1.4142 < \sqrt{2} < 1.4143,$$

or

$$\frac{14142}{10000} < \sqrt{2} < \frac{14143}{10000}.$$

We have here two rational numbers between which $\sqrt{2}$ lies. By going out to a sufficient number of decimals, we can obviously obtain two rational numbers containing $\sqrt{2}$ between them and differing from it by as little as we please. By taking successively 4, 5, 6, $\cdots$ decimals, proceeding as above and noting each time the smaller of the two rational numbers, we obtain a series or sequence of rational numbers which increase and approach $\sqrt{2}$; by noting each time the larger of the two numbers, we obtain a second sequence of rational numbers which decrease and also approach $\sqrt{2}$.

If on the other hand we consider the sequence of numbers

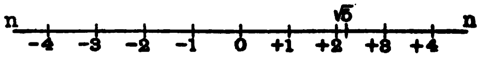
$$1, \frac{13}{10}, \frac{133}{100}, \frac{1333}{1000} \cdots$$

these evidently approach the value $\frac{4}{3}$, which is a rational number.

The idea here indicated is used to define irrational numbers. Without going further into the subject here, we shall say that *an irrational number is one which can be represented to any degree of approximation, but not exactly, as the quotient of two integers.*

31. Real Numbers. — The rational numbers, including all integers and quotients of integers, and the irrational numbers which do not involve an even root of a negative number, together constitute the class of *real numbers.*

It is often convenient to think of all the real numbers as represented by the points of a line.



Draw a line *nn* and mark on it a series of equidistant points. Call one of these points 0. Then mark the points to the right of zero in order, 1, 2, 3, · · · , and those to the left of zero, -1, -2, -3, · · · . In this way all integers are represented by points of the line *nn*, or by the distances of these points from the zero point.

Fractional and irrational numbers will be represented by intermediate points, as indicated by $\sqrt{5}$ in the figure.

In connection with the above diagram, the student may be interested to ponder the following statements regarding the system of real numbers.

(*a*) There is an unlimited number of integers, of rational numbers, and of irrational numbers.

(*b*) Between any two rational numbers there lies a third, and hence an unlimited number, of other rational numbers.

(*c*) Between any two rational numbers there lies an unlimited number of irrational numbers.

(*d*) There is no number which follows immediately after a given number. Geometrically stated, there is no point on a line immediately after a given point.

32. Irrational Expressions. — We now extend the idea of irrationality to algebraic quantities in general by the following definition:

An algebraic expression is said to be irrational when its parts are affected by other than the four fundamental operations.

Hence any expression involving indicated roots is irrational. As examples, we have

$$\sqrt[5]{1+x^2};\ (x^2-xy)^{-\frac{1}{3}}+(xy-y^2);\ \sqrt{\frac{1+2a+a^2}{1-a}}.$$

The last expression may be simplified. Thus,

$$\sqrt{\frac{1+2a+a^2}{1-a}}=\frac{\sqrt{1+2a+a^2}}{\sqrt{1-a}}=\frac{1+a}{\sqrt{1-a}}.$$

A **surd expression** is one involving an indicated root which cannot be exactly found.

A **surd number** is an indicated root of a number which cannot be exactly found.

33. Irrational Exponents. — What meaning shall we attach to the expression $2^{\sqrt{2}}$? Let a_1, a_2, a_3, $\cdots$ be a series of rational numbers approaching $\sqrt{2}$ in value. Then the quantity toward which the series of numbers 2^{a_1}, 2^{a_2}, 2^{a_3}, $\cdots$ approaches is $2^{\sqrt{2}}$. Similarly we obtain a meaning for a^x, when x is irrational.

We now *define* a^x as a symbol subject to the following laws:

I. $a^{-x} = \dfrac{1}{a^x}$; IV. $(a^x)^y = a^{xy}$ (*not* a^{x^y});

II. $a^x a^y = a^{x+y}$; V. $(ab)^x = a^x b^x$;

III. $a^x \div a^y = a^{x-y}$; VI. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$;

provided that the symbols a, b, x, y, a^x, b^x, a^y, b^y stand for real numbers.

34. Imaginary Numbers. — When $x^2 = 1$, we have obviously $x = \pm 1$. What is x when $x^2 = -1$? The answer cannot be a real number, since the square of every such number is positive. To obtain an answer to the question, we introduce a new number whose symbol is $\sqrt{-1}$, and which is defined as the number whose square is -1.

Since $\sqrt{-1}$ is not a real number, it is often called *imaginary* and denoted by i. Hence *the number* $i \equiv \sqrt{-1}$ *is defined by the equation* $i^2 = -1$.

We now *define* $\sqrt{-a}$ by the equation

I. $$\sqrt{-a} = i\sqrt{a}.$$

This is in accordance with our rules for exponents, since

$$\sqrt{-a} = \sqrt{a \times -1} = \sqrt{a}\sqrt{-1} = i\sqrt{a}.$$

Then the product $\sqrt{-a} \times \sqrt{-b}$ is determined by the equation,

II. $$\sqrt{-a} \times \sqrt{-b} = i\sqrt{a} \times i\sqrt{b} = i^2\sqrt{ab} = -\sqrt{ab}.$$

The results of the operations of algebra, applied to any number, are always expressible in the form $a + bi$, where a and b are real. Such a result may be considered as consisting of a real units and b imaginary units, $a \times 1 + b \times i$; it is called a *complex number*.

Two numbers of the forms $a + bi$ and $a - bi$ are called *conjugate complex numbers*.

When $a = 0$, the complex number $a + bi$ becomes bi, called a *pure imaginary*.

The rules for operating with complex numbers are in comformity with those for real numbers, and will be so used.

Calculation with imaginary numbers leads to many curious and interesting results. Thus Euler, in 1746, showed that $i^i = 0.2078+$.

Examples.

(*a*) $(1 + i)(1 - i) = 1 - i^2 = 1 - (-1) = 2.$

(*b*) $(2 + \sqrt{-3})(2 - \sqrt{-3}) = (2 + i\sqrt{3})(2 - i\sqrt{3}) = 4 - 3i^2 = 7.$

(*c*)
$$\begin{aligned}(1 + \sqrt{-3})^3 &= (1 + i\sqrt{3})^3 \\ &= 1 + 3(i\sqrt{3}) + 3(i\sqrt{3})^2 + (i\sqrt{3})^3 \\ &= 1 + i \cdot 3\sqrt{3} + 9i^2 + i^3 \cdot 3\sqrt{3} \\ &= 1 + i \cdot 3\sqrt{3} - 9 - i \cdot 3\sqrt{3} = -8.\end{aligned}$$

Example (*c*) shows that $(1 + \sqrt{-3})$ is a cube root of -8.

35. Exercises.

1. By cubing, show that each of the numbers, 2, $(-1 + \sqrt{-3})$, $(-1 - \sqrt{-3})$ is a cube root of 8.

2. Show that -2, $(1 + \sqrt{-3})$, and $(1 - \sqrt{-3})$ are cube roots of -8.

3. By raising 2, -2, $2i$, $-2i$ to the fourth power, show that each of them is a fourth root of 16.

4. By squaring, show that the two square roots of $2i$ are $\pm(1 + i)$.

5. By cubing, show that the three cube roots of i are $-i$, $\frac{1}{2}(i + \sqrt{3})$, and $\frac{1}{2}(i - \sqrt{3})$.

36. Reduction of Surds. — The expression $\sqrt[n]{a}$ is usually called a *radical*, $\sqrt{}$ being the *radical sign*, n the *index* of the radical and a the *radicand*. When the radicand is not a perfect nth power, the expression is a **surd.**

A surd is said to be in its *simplest form* when all factors of the radicand which are perfect powers of the same index as that of the radical have been taken out from the radical sign. Thus:

$$\sqrt[3]{\frac{8a^4b^5}{27c^6}} = \frac{2ab}{3c^2}\sqrt[3]{ab^2}.$$

Two surds are *similar* when they can be expressed with the same index and radicand. Otherwise they are *dissimilar*.

A *quadratic surd* is one whose index is 2.

The following theorems are used in the reduction of surd expressions.

Theorem 1. *The sum, difference, product and quotient of two dissimilar quadratic surds are always surds.*

Proof: Let the surds be $\sqrt{a}$ and $\sqrt{b}$. Since they are dissimilar, neither ab nor $a \div b$ can be a perfect square. Hence the product or quotient of the two surds is a surd.

Further, let c be a rational number, and assume that

$$\sqrt{a} \pm \sqrt{b} = c.$$

Squaring, $$a \pm 2\sqrt{ab} + b = c^2,$$

or $$\pm 2\sqrt{ab} = c^2 - a - b.$$

But a surd cannot equal a rational expression by definition. Hence the assumption is false, and the sum or difference of two surds is also a surd.

Theorem 2. *Given a relation of the form* $a + \sqrt{b} = c + \sqrt{d}\cdot$ *then* $a = c$ *and* $b = d$.

For, on transposing, we have $\sqrt{b} - \sqrt{d} = c - a$; hence if $b \neq d$, we have a surd equal to a rational number, which is impossible. Therefore $b = d$. Hence also $a = c$.

Corollary 1. *If* $a + \sqrt{b} = 0$, *then* $a = 0$ and $b = 0$.

Corollary 2. *If* $a + ib = 0$, *then* $a = 0$ *and* $b = 0$, *a and b being real numbers.*

Theorem 3. *The index of a surd may be multiplied by any number if at the same time the radicand be raised to the power indicated by this number.*

For, $$\sqrt[n]{a} = a^{\frac{1}{n}} = a^{\frac{m}{mn}} = \sqrt[mn]{a^m}.$$

In comparing several surds or surd expressions this rule is used to reduce them to surds with a common index. This is accomplished by writing all the surds with fractional exponents and then reducing the exponents to a common denominator.

37. *To rationalize the denominator of* $\dfrac{A}{\sqrt{a} + \sqrt{b}}\cdot$

Rule. — Multiply both sides of the fraction by $\sqrt{a} - \sqrt{b}$.

For, $$\frac{A(\sqrt{a} - \sqrt{b})}{(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})} = \frac{A(\sqrt{a} - \sqrt{b})}{a - b}.$$

38. *To obtain the square root of* $a + \sqrt{b}$.

Assume that $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$. To find x and y.

Squaring, $$a + \sqrt{b} = x + y + 2\sqrt{xy} = x + y + \sqrt{4xy}.$$

Hence $$a = x + y \text{ and } b = 4xy \quad \textbf{(36)}.$$

Then $$a^2 - b = x^2 - 2xy + y^2 = (x - y)^2,$$
or $$\pm\sqrt{a^2 - b} = x - y.$$
But $$a = x + y.$$
Therefore $x = \frac{1}{2}(a \pm \sqrt{a^2 - b})$ and $y = \frac{1}{2}(a \mp \sqrt{a^2 - b})$.

To obtain the square root of $a - \sqrt{b}$, *assume that* $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$.

39. Exercises.

Write the following with positive exponents and in simplest form:

1. $\dfrac{3a^0b^{-2}c^{-4}}{5a^{-1}b^{-4}c^{-5}}$.
2. $\left(\dfrac{m^{-3}n^2}{m^{-1}n^{-2}}\right)^{-3}$.
3. $\left(\dfrac{a^{-2} - b^{-2}}{a^{-1} + b^{-1}}\right)^{-3}$.
4. $\left(\dfrac{a^{-2} - b^{-2}}{a^{-4} - b^{-4}}\right)^{-2}$.
5. $\dfrac{a^{\frac{2}{3}}b^{-\frac{4}{5}}}{x^{-3}y^{-5}} \times \dfrac{a^{-\frac{2}{3}}b^{-\frac{1}{5}}}{x^5y^3}$.
6. $\left(\dfrac{a^{-\frac{4}{5}}}{x^{\frac{1}{3}}y^{-\frac{2}{3}}}\right)^{-15}$.
7. $\left(\dfrac{x^{\frac{1}{4}}}{a^{-\frac{1}{3}}b^{-\frac{1}{2}}}\right)^{-12}$.
8. $\left(\dfrac{x^{-\frac{2}{3}} - y^{-\frac{2}{3}}}{x^{-\frac{1}{3}} - y^{-\frac{1}{3}}}\right)^{-2}$.

Reduce to radicals with the same index:

9. $\sqrt{3}$ and $\sqrt[3]{4}$.
10. $\sqrt[6]{32}$ and $\sqrt[4]{5}$.
11. $\sqrt[3]{2}$ and $\sqrt[5]{3}$.
12. $\sqrt[6]{5}$ and $\sqrt[9]{25}$.
13. $\sqrt{3}$, $\sqrt[3]{2}$, and $\sqrt[4]{5}$.
14. $\sqrt{3}$, $\sqrt[5]{8}$, and $\sqrt[3]{4}$.
15. $\sqrt[3]{\frac{1}{2}}$, $\sqrt[4]{\frac{5}{8}}$, and $\sqrt{\frac{8}{5}}$.
16. $\sqrt[5]{\frac{2}{9}}$, $\sqrt{\frac{7}{10}}$, and $\sqrt[10]{\frac{3}{4}}$.
17. $\sqrt[7]{\frac{5}{7}}$, $\sqrt{\frac{2}{3}}$, and $\sqrt[14]{\frac{2}{11}}$.
18. $\sqrt[9]{0.3}$, $\sqrt[6]{\frac{3}{2}}$, and $\sqrt[21]{\frac{7}{9}}$.
19. $\sqrt{a}$, $\sqrt[4]{b}$, and $\sqrt[6]{c}$.
20. $\sqrt[7]{x^3}$ and $\sqrt[5]{x^2}$.
21. $\sqrt[3]{a^2}$, $\sqrt[4]{a^3}$, and $\sqrt[9]{a^5}$.
22. $\sqrt{\dfrac{x}{y^2}}$, $\sqrt{\dfrac{y^3}{z}}$, and $\sqrt[3]{\dfrac{z^2}{y^4}}$.
23. $\sqrt[6]{\dfrac{m^2}{n^3}}$, $\sqrt[5]{\dfrac{1}{y^4}}$, and $\sqrt[3]{\dfrac{n^2}{y^4}}$.
24. $\sqrt[7]{\dfrac{x^2}{y}}$, $\sqrt{\dfrac{z}{y^2}}$, and $\sqrt[3]{\dfrac{3y^4}{z^5}}$.
25. $\sqrt[m]{\dfrac{1}{x^2}}$, $\sqrt[n]{x}$, and $\sqrt[p]{x^{-3}}$.

Combine by performing the indicated additions and subtractions, reducing to similar surds when necessary:

26. $4\sqrt{3} - 7\sqrt{3} + 9\sqrt{3}$.
27. $5\sqrt[3]{4} - 8\sqrt[3]{4} + 2\sqrt[3]{4}$.
28. $3\sqrt{2} + \sqrt{50}$.
29. $\sqrt{2} + 3\sqrt{18} - \frac{1}{2}\sqrt{72}$.
30. $5\sqrt[3]{4} + 3\sqrt[3]{32} - 2\sqrt[3]{108}$.
31. $\frac{1}{2}\sqrt[3]{5} + \frac{7}{2}\sqrt[3]{5} + \frac{3}{4}\sqrt[3]{40}$.
32. $3\sqrt{175} - 7\sqrt{63} + 5\sqrt{28}$.
33. $3\sqrt{a} + b\sqrt{a} - \sqrt{a^5}$.
34. $\sqrt[3]{27c^4} - \sqrt[3]{8c^4} + \sqrt[6]{64c^8}$.
35. $\sqrt{a^2x} - \sqrt[4]{a^8x^6} + \sqrt{a^4x^5}$.
36. $\sqrt[3]{a^3b^3c} + \sqrt[6]{a^{12}b^{12}c^8} - ab\sqrt[3]{c^7}$.

Reduce by the method of (**38**):

37. $\sqrt{6 + 2\sqrt{5}}$.
38. $\sqrt{11 + 6\sqrt{2}}$.
39. $\sqrt{7 + 4\sqrt{3}}$.
40. $\sqrt{8 - \sqrt{15}}$.

41. $\sqrt{5+\sqrt{21}}.$

42. $\sqrt{10-2\sqrt{21}}.$

43. $\sqrt{7-2\sqrt{10}}.$

44. $\sqrt{7+4\sqrt{3}}.$

45. $\sqrt{13+2\sqrt{30}}.$

46. $\sqrt{16-6\sqrt{7}}.$

Perform the following multiplications and divisions:

47. $(5+2\sqrt{2})\,(5-2\sqrt{2}).$

48. $(5+3\sqrt{3})\,(3-5\sqrt{3}).$

49. $(2\sqrt{6}-3\sqrt{3})\,(\sqrt{3}+2\sqrt{2}).$

50. $(\sqrt{10}-\sqrt{3})\,(\sqrt{2}+\sqrt{5}).$

51. $(\sqrt[3]{9}-3\sqrt[3]{4})\,(4\sqrt[3]{3}+\sqrt[3]{2}).$

52. $(\sqrt{a+b}+\sqrt{a})\,(\sqrt{a+b}-\sqrt{a}).$

53. $\sqrt{m+\sqrt{n}}\,\sqrt{m-\sqrt{n}}.$

54. $\sqrt{a^2-1}\times\sqrt{\dfrac{a+1}{a-1}}.$

55. $\sqrt{a\sqrt[3]{a^4}}\times\sqrt{\sqrt[3]{a}}.$

56. $\sqrt[3]{x\sqrt{x^6}}\times\sqrt[3]{x\sqrt[3]{x^8}}.$

57. $\sqrt[3]{x^4y^2}\times\sqrt[5]{x^4y^9}.$

58. $\sqrt[4]{\dfrac{ab^2}{x^7}}\times\sqrt[6]{\dfrac{cx^5}{a^2b}}.$

59. $\sqrt{28}\div\sqrt{4}.$

60. $\sqrt{48}\div\sqrt{6}.$

61. $\sqrt{32}\div\sqrt{8}.$

62. $\sqrt[3]{40}\div\sqrt[3]{5}.$

63. $\sqrt[4]{324}\div\sqrt[4]{4}.$

64. $\sqrt[3]{12}\div\sqrt{4}.$

65. $\sqrt{54}\div\sqrt[4]{72}.$

66. $\sqrt{x^3}\div\sqrt[9]{x^7}.$

67. $\sqrt[5]{a^4}\div\sqrt[7]{2a^2}.$

68. $\sqrt[11]{a^5b^9}\div\sqrt[7]{a^3b^4}.$

69. $\sqrt[6]{27^{-1}z^9}\div\sqrt{6z^7}.$

70. $\sqrt[4]{8x^3y^2}\div\sqrt[3]{2x^2y^4}.$

Express with fractional exponents instead of radicals:

71. $(\sqrt[3]{m^2})^4.$

72. $(\sqrt{n^3})^5.$

73. $(\sqrt[4]{a^5})^7.$

74. $(\sqrt[8]{x^2y^3})^5.$

75. $(-\sqrt[3]{a^2})^9.$

76. $(-\sqrt{a^3})^{10}.$

77. $(\sqrt[3]{\sqrt{a^5}})^5.$

78. $(\sqrt[5]{\sqrt[4]{16a^{17}}})^{20}.$

79. $(\sqrt[4]{(x+y)^3})^6.$

80. $(\sqrt[r]{\sqrt[s]{x^my^n}})^t.$

Rationalize the denominator of:

81. $\dfrac{1}{\sqrt{2}}.$

82. $\dfrac{a}{\sqrt{b}}.$

83. $\dfrac{a}{\sqrt[3]{a^2}}.$

84. $\dfrac{a}{2+\sqrt{a}}.$

85. $\dfrac{a}{2-\sqrt{a}}.$

86. $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}.$

87. $\dfrac{3-2\sqrt{a^3}}{2\sqrt{a^3}+3}.$

88. $\dfrac{1}{\sqrt{x+y}+\sqrt{x-y}}.$

89. $\dfrac{\sqrt{8}-\sqrt{7}}{\sqrt{7}+\sqrt{2}}.$

90. $\dfrac{\sqrt{13}+\sqrt{10}}{\sqrt{10}-\sqrt{13}}.$

91. $\dfrac{1}{a\sqrt{b}-c\sqrt{d}}.$

92. Calculate to three decimal places the values of the fractions in exercises 89 and 90.

Perform the following operations and simplify results:

93. $\sqrt[5]{a\sqrt[3]{a^2}} \times \sqrt[6]{\sqrt[4]{a^9}} \times \sqrt[3]{a^2\sqrt[8]{a^{15}}}$.

94. $(x^{-1} + x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1})\,(x^{-1} - x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1})$.

95. $(2\,a^{-\frac{3}{4}} - 3\,a^{-\frac{2}{4}} - a^{-\frac{1}{4}} - 2)\,(a^{-\frac{2}{4}} - 2\,a^{-\frac{1}{4}} + 3)$.

96. Write out the result of replacing $a^{-\frac{1}{4}}$ by b in exercise 95.

97. $\left(a^{\frac{3}{x}} - 2\,a^{\frac{2}{x}} - 3\,a^{\frac{1}{x}}\right)\left(2\,a^{\frac{2}{x}} - a^{\frac{1}{x}} + 2\right)$.

98. $\left(y^{\frac{3}{n}} - ay^{\frac{2}{n}} + 3\,by^{\frac{1}{n}} - c\right)\left(y^{\frac{2}{n}} - by^{\frac{1}{n}} - cy^0\right)$.

99. $(2\,a^{-2}b^{-\frac{2}{3}} - 3\,a^{-\frac{2}{3}}b^{-\frac{4}{3}})^2$.

100. $(a^{-\frac{1}{2}} + b^{-\frac{1}{2}})^3$.

101. $(x^{\frac{2}{3}} - y^{\frac{2}{3}})^3$.

102. $(1^{-3} + n^{-\frac{2}{3}})^4$.

103. $(m^{-2} + m^{-\frac{1}{2}})^4$.

104. $(a^{-1}x^{-2} - ax^2)^4$.

105. $(a^{\frac{3}{2}} + a^{\frac{3}{4}} - a^{\frac{1}{2}})^2$.

106. $(2\,a^{\frac{3}{2}} - 3\,b^{\frac{2}{3}} - c^{\frac{1}{3}})^2$.

107. $(a^{\frac{1}{2}} - 2\,b^{\frac{1}{3}} + 3\,c^{\frac{1}{4}} - 2\,d^{\frac{1}{5}})^2$.

108. $(x^{\frac{4}{3}}y^{\frac{1}{2}} - 3\,x^{\frac{2}{3}}y^{\frac{3}{2}} + 3\,x^{\frac{2}{3}}y - 2\,x^{\frac{1}{3}}y^{\frac{3}{2}})^2$.

109. Write out the result of replacing $x^{\frac{1}{3}}$ by u and $y^{\frac{1}{2}}$ by v in exercise 108.

110. $(x + 1) \div (\sqrt[3]{x} + 1)$.

111. $(\sqrt[5]{x^3} + 1) \div (\sqrt[5]{x} + 1)$.

112. $(\sqrt{x} - \sqrt{y}) \div (\sqrt[4]{x} + \sqrt[4]{y})$.

113. $(a^{\frac{1}{2}} - b^{\frac{1}{2}}) \div (a^{\frac{1}{16}} - b^{\frac{1}{16}})$.

114. $(x^{\frac{3}{2}} - xy^{\frac{1}{2}} - x^{\frac{1}{2}}y + y^{\frac{3}{2}}) \div (\sqrt{x} - \sqrt{y})$.

115. $(a^{\frac{5}{2}} - a^2 - 4\,a^{\frac{3}{2}} + 6\,a - 2\sqrt{a}) \div (a^{\frac{3}{2}} - 4\sqrt{a} + 2)$.

116. $(a^{\frac{1}{2}} - b^{\frac{1}{2}} - c^{\frac{1}{2}} + 2\sqrt[4]{bc})\,(a^{\frac{1}{4}} + b^{\frac{1}{4}} - c^{\frac{1}{4}})$.

Express the following in the form $a\sqrt{-1}$.

117. $\sqrt{-9}$; $\sqrt{-25}$; $\sqrt{-81}$.

118. $\sqrt{-25} - \sqrt{-49} + \sqrt{-121}$.

119. $\sqrt{-a^2}$; $\sqrt{-b^2}$; $\sqrt{-x^{2n}}$.

120. $\sqrt{-a^4} + \sqrt{-a^2} - \sqrt{-4\,a^4}$.

121. $\sqrt{-(m+n)^2} - \sqrt{-(m-n)^2} - \sqrt{-n^2}$.

Verify by raising both sides to the same power:

122. $\sqrt[4]{4096} = \pm 8$ or $\pm 8\,i$.

123. $\sqrt[3]{-216} = -6$ or $3\,(1 \pm i\sqrt{3})$.

124. $\sqrt{1+i} = \frac{1}{2}\left(\sqrt{2 \pm 2\sqrt{2}} + i\sqrt{-2 \pm 2\sqrt{2}}\right)$.

125. $\sqrt[3]{1+i} = \frac{1}{2}\sqrt[3]{4}\,(i - 1)$, or $\frac{1}{4}\sqrt[3]{4}\,[(\pm\sqrt{3} + 1) + i\,(\pm\sqrt{3} - 1)]$.

Multiply and reduce to the form $a + bi$: $(i \equiv \sqrt{-1})$.

126. $(a + b\sqrt{-1})(a - b\sqrt{-1})$.

127. $(3 + 5\,i)\,(4 + 7\,i)$.

128. $(x - 2\,i)\,(y - 3\,i)$.

129. $(\sqrt{8} + i\sqrt{12})(\sqrt{2} + i\sqrt{3})$.

130. $(-1 + i\sqrt{3})^4$.

131. $\left(\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}\right)^4$.

Reduce to the form $a + bi$ by rationalizing the denominator:

132. $\dfrac{2}{3 - 2\,i}$

133. $\dfrac{1+i}{1-i}$.

134. $\dfrac{a - bi}{a + bi}$.

135. $\dfrac{a + i\sqrt{x}}{a - i\sqrt{x}}$.

136. $\dfrac{6}{7 - 2\sqrt{-5}}$.

137. $\dfrac{1 - i^3}{(1 - i)^3}$.

Clear the following equations of radicals or of fractional exponents:

Example. To clear the equation $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ of radicals put

$$\sqrt{x} + \sqrt{y} = 1 - \sqrt{z};$$

squaring, $x + y + 2\sqrt{xy} = 1 + z - 2\sqrt{z},$

or $x + y - z - 1 = -2\sqrt{xy} - 2\sqrt{z}.$

Squaring again, $(x + y - z - 1)^2 = 4xy + 4z + 8\sqrt{xyz},$

or $(x + y - z - 1)^2 - 4(xy + z) = 8\sqrt{xyz}.$

Squaring again, $[(x + y - z - 1)^2 - 4(xy + z)]^2 = 64xyz.$ q.e.f.

138. $\sqrt{x + 4} = 4.$

139. $\sqrt{2x + 6} = 3.$

140. $\sqrt[3]{x + 1} = 2.$

141. $\sqrt[4]{ax + b} = c.$

142. $x^{\frac{1}{3}} + y^{\frac{1}{3}} = 1.$

143. $\sqrt{x + 1} - \sqrt{x - 1} = 2.$

144. $\sqrt{32 + x} = 16 - \sqrt{x}.$

145. $\sqrt{x + 20} - \sqrt{x - 1} - 3 = 0.$

146. $\sqrt{x} - \dfrac{5}{\sqrt{x}} = \sqrt{x - 9}.$

147. $\sqrt{15 + \sqrt{2x + 80}} = 5.$

148. $\sqrt{3 + \sqrt{x}} = \sqrt{12 - 2\sqrt{x}}.$

149. $\sqrt{3 + \sqrt{x}} = \sqrt{11 - 3\sqrt{x}}.$

150. $x^{\frac{2}{3}} + x^{\frac{1}{3}} = 1.$

CHAPTER III

LOGARITHMS. BINOMIAL THEOREM FOR POSITIVE INTEGRAL EXPONENTS

40. Logarithm. — The simple laws of operation for exponents have given rise to a method of calculation involving the use of a function called the *logarithm*. We shall first illustrate this method.

Suppose that we know the powers of 10 which are required to produce a set of numbers, as in the adjacent table, where the exponents are given to the nearest figure in the third decimal. *The exponent of* 10 *in each equation is called the common logarithm* (*or the logarithm to the base* 10) *of the number on the left.* Thus, the logarithm of 5.00 is 0.699, of 5.50 is 0.740, and so on. As equations, we write

$$\log_{10} 5.00 = 0.699,$$
$$\log_{10} 5.50 = 0.740,$$

and so on.

TABLE.

$$5.00 = 10^{0.699}$$
$$5.50 = 10^{0.740}$$
$$6.00 = 10^{0.778}$$
$$6.50 = 10^{0.813}$$
$$7.00 = 10^{0.845}$$
$$7.50 = 10^{0.875}$$
$$8.00 = 10^{0.903}$$
$$8.50 = 10^{0.929}$$
$$9.00 = 10^{0.954}$$
$$9.50 = 10^{0.978}$$
$$10.00 = 10^{1.000}$$

41. Use of Logarithms. — By aid of such a table *products* of numbers (within certain limits) can be obtained by *adding* the logarithms of the factors; also, *division* is reduced to *subtraction* of logarithms.

Example 1. Find the value of $6.5 \times 8.5 \times 9.5$.

We have
$$\begin{aligned} 6.5 \times 8.5 \times 9.5 &= 10^{0.813} \times 10^{0.929} \times 10^{0.978} \\ &= 10^{0.813+0.929+0.978} \\ &= 10^{2.720} = 10^{2} \times 10^{0.720}. \end{aligned}$$

Now 0.720 lies almost exactly midway between 0.699 and 0.740; hence we assume that the number corresponding to $10^{0.720}$ will be midway between 5.00 and 5.50 and is equal to 5.25. (The process here used is called **interpolation.** It involves the assumption that a logarithm changes proportionately to the change in the number, an assumption which is not exactly, but very nearly, true except for numbers near zero, provided the changes in the numbers are small.)

Therefore

$$6.5 \times 8.5 \times 9.5 = 10^{2} \times 10^{0.720} = 100 \times 5.25 = 525.$$

The exact value is 524.875.

Example 2. — Find the value of $\frac{6.25 \times 7.20}{5.75}$.

Let $10^a = 6.25;\quad 10^b = 7.20;\quad 10^c = 5.75.$

Then $\frac{6.25 \times 7.20}{5.75} = \frac{10^a \times 10^b}{10^c} = 10^{a+b-c}.$

Since 6.25 lies halfway between 6.00 and 6.50, we take for a the value halfway between the corresponding exponents, so that $a = 0.795$ (more exactly 0.7955). To get b, we note that 7.20 lies $\frac{2}{5}$ of the way from 7.00 to 7.50; hence we take for b the number lying in the corresponding position between the exponents 0.845 and 0.875. Therefore

$$b = 0.845 + \tfrac{2}{5} \times 0.030 = 0.857.$$

Similarly $c = 0.759.$

Hence $\frac{6.25 \times 7.20}{5.75} = 10^{0.795+0.857-0.759} = 10^{0.893}.$

The corresponding number lies between 7.50 and 8.00, and nearer the latter. Since our exponent, 0.893, lies $\frac{18}{28}$ of the way from 0.875 to 0.903, we find the number lying in the corresponding position between 7.50 and 8.00, that is,

$$7.50 + \tfrac{18}{28} \times 0.50 = 7.50 + 0.32 = 7.82.$$

Therefore $\frac{6.25 \times 7.20}{5.75} = 7.82$, approximately.

This result is correct to two decimals. More exactly it is 7.8261.

By the aid of our table, powers and roots of numbers may be found by applying the operations of multiplication and division, respectively, to their logarithms.

Example. Find the value of $\sqrt[4]{9.35^3}$.

We have $\sqrt[4]{9.35^3} = (9.35)^{\frac{3}{4}}.$

Let $9.35 = 10^a;\quad$ then $(9.35)^{\frac{3}{4}} = 10^{\frac{3}{4}a}.$

From the table, $a = 0.954 + \tfrac{7}{10} \times 0.024 = 0.971.$

Therefore $\sqrt[4]{9.35^3} = 10^{0.728} = 5.00 + \tfrac{29}{41} \times 0.50 = 5.35.$

A more accurate value is 5.335, so that the second decimal of our result is slightly in error.

Exercise. Using the above table, calculate value of:

$$7.75 \times 9.35 \times 8.05; \quad (5.15 \times 9.38) \div 7.62; \quad \sqrt[5]{8.67^6}.$$

Obviously the calculation of the last result by the methods of arithmetic would be very tedious, and with a slight increase in the complexity of the exponent these methods would become quite useless.

We shall now consider the general theory of the method illustrated above.

42. Logarithm of a Number. — Let a be a certain fixed number, n any other number, and let x be the exponent of a required to produce n. *Then x is the logarithm of n to the base a.*

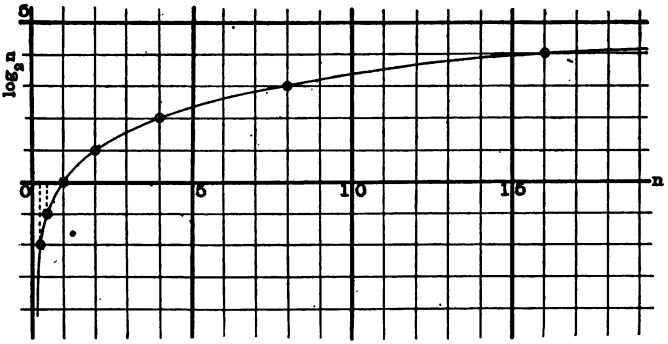
As equations,

if $a^x = n$, *then* $x = \log_a n$.

We give below some very simple *tables of logarithms.*

Number.	Logarithm Base = 2.	n.	$\log_{10} n$.	n.	$\log_{10} n$.
$\frac{1}{8}$	− 3	.001	− 3	5.00	0.699
$\frac{1}{4}$	− 2	.01	− 2	5.50	0.740
$\frac{1}{2}$	− 1	.1	− 1	6.00	0.778
1	0	1.0	0	6.50	0.813
2	1	10	1	7.00	0.845
4	2	100	2	7.50	0.875
8	3	1000	3	8.00	0.903

It will be instructive to represent graphically a table of logarithms, as in the following figure. This is constructed as follows, from the data given in the first of the above three tables of logarithms.



On line *On* mark a number scale, choosing any convenient length to represent 1. Now represent the logarithm of a number n by a line whose length shall be $\log_2 n$, and which is drawn at right angles to the number scale, starting from the point n. Thus to represent $\log_2 4$, which equals 2 by the table, start at 4 on *On* and erect a perpendicular 2 units long. When the logarithm is *negative,* the perpendicular is drawn *downward.* Thus, to represent $\log_2 \frac{1}{4}$

start at $\frac{1}{4}$ on *On* and draw a perpendicular 2 units long and downward.

Construct in this way perpendiculars to represent the logarithms of several numbers; then draw a smooth curve through the free ends of these perpendiculars. This curve represents graphically the logarithms of numbers to the base 2. From it we can read off approximately the values of the logarithm of any number falling within the range of the diagram.

Thus: $\log_2 1.5 = 0.6$; $\log_2 3.2 = 1.8$.

Exercise 1. From the figure read off the values of

$\log_2 1.7$; $\log_2 2.5$; $\log_2 4.3$; $\log_2 0.7$.

Exercise 2. Construct a table and curve for $\log_3 n$; for $\log_4 n$; for $\log_{10} n$. (In these diagrams it will be convenient to shorten the number scale by taking one square to represent more than 1 unit; thus for $\log_4 n$ take one square on *On* as 4 units. The vertical scale may still be used with one square as one unit.)

Exercise 3. Construct a table and curve for $\log_{\frac{1}{2}} n$; for $\log_{\frac{1}{3}} n$. (Here use a longer scale on *On*.)

43. Exercises.

1. What is the value of $\log_a 1$?
2. What are the logarithms of 8, 16, 64, 128, to the base 2?
3. What are the logarithms of 8, 16, 64, 128 to the base $\frac{1}{2}$?
4. What are the logarithms of $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{243}$, to the base of 3? to the base $\frac{1}{3}$?
5. What are the logarithms of $\frac{125}{8}$ and $\frac{625}{16}$ to the base $\frac{2}{5}$?
6. What are the logarithms of 2, 4, 8 to the base 16?
7. What is the base, if $\log 2 = 1$? if $\log a = 1$?
8. What is the base, if $\log \frac{1}{3} = 4$? if $\log 25 = -2$?
9. What is the base, if $\log 49 = 2$? if $\log 0.0081 = 4$?
10. What can you say about the logarithm of a negative number when the base is positive?
11. Why would it be inconvenient to use a negative number as the base of a system of logarithms?
12. If $n = (e^{x+y})^{x-y}$, find $\log_e n$.
13. If $x = \sqrt[p]{e}\,\sqrt[q]{e}\,(e^{p+q})$, find $\log_e x$.
14. If $a = \left[(10^{x^3-y^3})^{\frac{1}{x-y}}\right]^{x^2-xy+y^2}$, find $\log_{10} a$.
15. Show that $a^{\log_a x} = x$.
16. Show that $x + \log_e (1 + e^{-2x}) = \log_e (e^x + e^{-x})$. Hint: $1 + e^{-2x} = \dfrac{e^x + e^{-x}}{e^x}$.
17. Find a if $\log_a 3 = 3$; if $\log_a m = m$. *Ans.* $\sqrt[3]{3}$; $\sqrt[m]{m}$.

44. Laws of Operation with Logarithms. — Since a *logarithm is an exponent,* the laws of operation for logarithms are the same as those for exponents. They are as follows:

I. *The logarithm of a product equals the sum of the logarithms of the factors.*

II. *The logarithm of a fraction equals the logarithm of the numerator minus the logarithm of the denominator.*

III. *The logarithm of any power of a number equals the exponent of the power times the logarithm of the number.*

IV. *The logarithm of any root of a number equals the logarithm of the number divided by the index of the root.*

Proofs. Let x be the logarithm of m, y that of n, the base being a. Then

$$\begin{cases} \log_a m = x, \\ \log_a n = y, \end{cases} \quad \text{or} \quad \begin{cases} a^x = m, \\ a^y = n. \end{cases}$$

Hence
$$mn = a^{x+y} \text{ and } \frac{m}{n} = a^{x-y}.$$

That is,
$$\log_a mn = x + y = \log_a m + \log_a n,$$

and
$$\log_a \frac{m}{n} = x - y = \log_a m - \log_a n.$$

We have therefore the rules I and II.

Again, let $\log_a m = x,$ so that $m = a^x.$

Then, if p and q be any real numbers,

$$m^p = a^{px} \text{ and } \sqrt[q]{m} = a^{\frac{x}{q}}.$$

Hence
$$\log_a m^p = px = p \log_a m,$$

and
$$\log_a \sqrt[q]{m} = \frac{x}{q} = \frac{1}{q}\log_a m.$$

We have therefore rules III and IV.

(Rule III contains rule IV, since the power in question may be fractional.)

The following facts regarding logarithms should also be carefully noted.

(*a*) In any system the logarithm of the base is 1.

For $a^1 = a. \quad \therefore \log_a a = 1.$

(*b*) In any system the logarithm of 1 is 0.

For $a^0 = 1. \quad \therefore \log_a 1 = 0.$

(*c*) In any system whose base is greater than unity, the logarithm of 0 is $-\infty$.

For if $a^x = m$ and $a > 1$, then if x is a large negative number, m will be small. As x increases indefinitely, always being negative, m approaches zero. That is,

$$a^{-\infty} = 0 \text{ if } a > 1; \qquad \therefore \log 0 = -\infty.$$

(*d*) A negative number has no (real) logarithm, the base being positive.

(*e*) As a number varies from 0 to $+\infty$, its logarithm varies from $-\infty$ to $+\infty$, the base being greater than 1.

When the number is greater than 1, its logarithm is positive.

When the number is less than 1, its logarithm is negative.

Exercises. — Prove:

1. $\log_a mnr = \log_a m + \log_a n + \log_a r.$
2. $\log_a \frac{mn}{rs} = \log_a m + \log_a n - \log_a r - \log_a s.$
3. $\log_a m^p n^q = p \log_a m + q \log_a n.$
4. $\log_a \frac{m^p}{n^q} = p \log_a m - q \log_a n.$
5. $\log_a \sqrt{m^3 n^5} = \frac{3}{2} \log_a m + \frac{5}{2} \log_a n.$
6. $\log_a \sqrt[3]{\frac{m^2 n}{r^4}} = \frac{2}{3} \log_a m + \frac{1}{3} \log_a n - \frac{4}{3} \log_a r.$
7. $\log \frac{(mn)^3}{\sqrt{r^5 s^3}} = 3 \log_a m + 3 \log_a n - \frac{5}{2} \log_a r - \frac{3}{2} \log_a s.$
8. $\log_a \sqrt[3]{3 \sqrt[3]{3}} = \frac{4}{9} \log_a 3.$

45. Table of Common Logarithms. — The *common logarithm* of a number is its logarithm to the base 10.

The table on pages 40, 41 gives the decimal parts of such logarithms, to four places of decimals, for any number containing not more than three significant figures. The integral part of the logarithm must be supplied by the computer, as explained below.

The decimal part of a logarithm is called the *mantissa*.

The integral part of a logarithm is called the *characteristic*.

Rules for Characteristics.

(**a**) When the number has n significant figures to the left of the decimal point, the characteristic of its logarithm is $n - 1$.

(**b**) When the number is a decimal with n ciphers between the decimal point and the first digit which is not zero, the characteristic of its logarithm is $9 - n$, and -10 must be supplied to complete the logarithm.

The reason for these rules will become evident when we consider an example. We shall now write $\log_{10} n$ simply $\log n$.

Example. Let us find log 302. In the table find 30 in the left-hand column and run across the page horizontally to the column headed 2. There we find that

$$\text{mantissa of } \log 302 = .4800.$$

Now 302 lies between 100 and 1000, i.e. between 10^2 and 10^3. Hence, by the definition of a logarithm, log 302 must lie between 2 and 3. Therefore the characteristic is 2, and

$$\log 302 = 2.4800.$$

This is of course not the *exact* logarithm of 302, but only its value to four decimal places.

Writing the last equation in exponential form, we have

$$302 = 10^{2.4800}.$$

Multiplying both sides by 10,

$$3020 = 10 \times 10^{2.4800} = 10^{3.4800}. \quad \text{Hence } \log 3020 = 3.4800.$$

Multiplying again by 10,

$$30200 = 10 \times 10^{3.4800} = 10^{4.4800}. \quad \text{Hence } \log 30200 = 4.4800.$$

Therefore, where a number is multiplied by 10, the characteristic of its logarithm is increased by 1; the mantissa remains unchanged.

Dividing the above equation successively by 10, we obtain

$$\begin{aligned} 30.2 &= 10^{2.4800} \div 10 = 10^{1.4800}, \\ 3.02 &= 10^{1.4800} \div 10 = 10^{0.4800}, \\ .302 &= 10^{0.4800} \div 10 = 10^{0.4800-1}, \\ .0302 &= 10^{0.4800-1} \div 10 = 10^{0.4800-2}, \\ .00302 &= 10^{0.4800-2} \div 10 = 10^{0.4800-3}, \end{aligned}$$

and so on. As logarithmic equations these are:

$$\begin{aligned} \log 30.2 &= 1.4800, \\ \log 3.02 &= 0.4800, \\ \log .302 &= 0.4800 - 1 = 9.4800 - 10, \\ \log .0302 &= 0.4800 - 2 = 8.4800 - 10, \\ \log .00302 &= 0.4800 - 3 = 7.4800 - 10, \end{aligned}$$

and so on. The second form in the last three equations is used for convenience in computations; it is in accordance with rule (**b**).

To discuss rules (**a**) and (**b**) more generally, let n be any number. Then by the definition of a logarithm, when

	m lies between	$\log m$ lies between
(1)	1 and 10,	0 and 1,
(2)	10 and 100,	1 and 2,
(3)	100 and 1000,	2 and 3,
(4)	1000 and 10000,	3 and 4,

and so on. Therefore, when m has

(1) 1 digit to the left of the point, $\log m = 0. + \cdots$;
(2) 2 digits to the left of the point, $\log m = 1. + \cdots$;
(3) 3 digits to the left of the point, $\log m = 2. + \cdots$;
(4) 4 digits to the left of the point, $\log m = 3. + \cdots$;

and so on. Hence rule (**a**).

In the case of decimal numbers, when

	m lies between	$\log m$ lies between
(1)	1 and 0.1,	0 and -1,
(2)	0.1 and 0.01,	-1 and -2,
(3)	0.01 and 0.001,	-2 and -3,
(4)	0.001 and 0.0001,	-3 and -4,

and so on. That is, when m is a decimal number in which

(1) no cipher follows the point, $\log m = 9. + \cdots - 10$;
(2) 1 cipher follows the point, $\log m = 8. + \cdots - 10$;
(3) 2 ciphers follow the point, $\log m = 7. + \cdots - 10$;
(4) 3 ciphers follow the point, $\log m = 6. + \cdots - 10$;

and so on. Hence rule (**b**).

Interpolation. — *Example.* Find log 3024.

From the table,

$$\begin{aligned} &\text{mantissa of } \log 302 = .4800; \\ &\text{mantissa of } \log 303 = .4814; \end{aligned} \quad \text{difference} = .0014.$$

Assuming that the increase in the logarithm is proportional to the increase in the number, we have

$$\text{mantissa of } \log 3024 = .4800 + .4 \times .0014 = .4806.$$

The result is here given to the nearest unit in the fourth decimal place, $.4 \times .0014$ being taken equal to .0006 in place of .00056.

Proportional Parts. — For convenience in interpolation, the tabular differences greater than 20 are subdivided into tenths and tabulated under the heading "Prop. Parts." When the difference is less than 20, the interpolation is best made mentally. If it is desired, the table of proportional parts may be used when $d < 20$ by taking half the proportional part corresponding to double the difference.

Examples.

1. $\log 164.3 = ?$

$$\begin{aligned} \text{Mantissa of } \log 164 &= .2148; \quad d = 27, \\ \text{Correction for } .3 &= \underline{8} \\ \log 164.3 &= 2.2156 \end{aligned}$$

2. $\log (164.3)^{\frac{2}{3}} = ?$

$$\begin{aligned} \log (164.3)^{\frac{2}{3}} &= \tfrac{2}{3} \log 164.3 \\ &= \tfrac{2}{3} (2.2156) = 1.4771. \end{aligned}$$

3. $\log .01047 = ?$

$$\begin{aligned} \text{Mantissa of } \log 104 &= .0170; \quad d = 42, \\ \text{Correction for } .7 &= \underline{29} \\ \log .01047 &= 8.0199 - 10 \end{aligned}$$

4. $\log \sqrt[3]{(.01047)^4} = ?$

$$\begin{aligned} \sqrt[3]{.01047^4} &= (.01047)^{\frac{4}{3}}, \\ \log \sqrt[3]{(.01047)^4} &= \tfrac{4}{3} \log (.01047) \\ &= \tfrac{4}{3} (8.0199 - 10). \\ 4\,(8.0199 - 10) &= 32.0796 - 40 = 22.0796 - 30. \\ \tfrac{1}{3}\,(22.0796 - 30) &= 7.3599 - 10. \end{aligned}$$

Note. When a logarithm which is followed by -10 is to be divided by a number, add and subtract a multiple of ten so that the quotient will come out in a form followed by -10. Thus:

$$\tfrac{1}{4}\,(8.2448 - 10) = \tfrac{1}{4}\,(38.2448 - 40) = 9.5612 - 10.$$

Anti-logarithm. — The number whose logarithm is x is called the *anti-logarithm* of x.

Thus, if $x = \log m$, then $m = \text{anti-log}\, x$.

Given a logarithm, to obtain the corresponding number (anti-logarithm).

Examples.

1. $\log m = 0.4806. \quad m = ?$

The given logarithm lies between the tabular logarithms .4800 and .4814, to which correspond the numbers 302 and 303 respectively. Thus we have

$$\begin{array}{cc} \text{Number.} & \text{Mantissa of log.} \\ \begin{array}{c} 302 \\ m \\ 303 \end{array} & \left.\begin{array}{l} \left.\begin{array}{l} .4800 \\ .4806 \end{array}\right\} 6 \\ .4814 \end{array}\right\} 14 \end{array}$$

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
No.	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	43	42
1	4.3	4.2
2	8.6	8.4
3	12.9	12.6
4	17.2	16.8
5	21.5	21.0
6	25.8	25.2
7	30.1	29.4
8	34.4	33.6
9	38.7	37.8

	41	40
1	4.1	4.0
2	8.2	8.0
3	12.3	12.0
4	16.4	16.0
5	20.5	20.0
6	24.6	24.0
7	28.7	28.0
8	32.8	32.0
9	36.9	36.0

	39	38
1	3.9	3.8
2	7.8	7.6
3	11.7	11.4
4	15.6	15.2
5	19.5	19.0
6	23.4	22.8
7	27 3	26.6
8	31.2	30.4
9	35.1	34.2

	37	36
1	3.7	3.6
2	7.4	7.2
3	11.1	10.8
4	14.8	14.4
5	18.5	18.0
6	22.2	21.6
7	25.9	25.2
8	29.6	28.8
9	33.3	32.4

	35	34
1	3.5	3.4
2	7.0	6.8
3	10.5	10.2
4	14.0	13.6
5	17.5	17.0
6	21.0	20.4
7	24.5	23.8
8	28.0	27.2
9	31.5	30.6

	33	32
1	3.3	3.2
2	6.6	6.4
3	9.9	9.6
6	13.2	12.8
5	16.5	16.0
6	19.8	19.2
7	23.1	22.4
8	26.4	25.6
9	29.7	28.8

Prop. Parts

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
No.	0	1	2	3	4	5	6	7	8	9

Prop. Parts		
	31	**30**
1	3.1	3.0
2	6.2	6.0
3	9.3	9.0
4	12.4	12.0
5	15.5	15.0
6	18.6	18.0
7	21.7	21.0
8	24.8	24.0
9	27.9	27.0
	29	**28**
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2
	27	**26**
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4
	25	**24**
1	2.5	2.4
2	5.0	4.8
3	7.5	7.2
4	10.0	9.6
5	12.5	12.0
6	15.0	14.4
7	17.5	16.8
8	20.0	19.2
9	22.5	21.6
	23	**22**
1	2.3	2.2
2	4.6	4.4
3	6.9	6.6
4	9.2	8.8
5	11.5	11.0
6	13.8	13.2
7	16.1	15.4
8	18.4	17.6
9	20.7	19.8
	21	
1	2.1	
2	4.2	
3	6.3	
4	8.4	
5	10.5	
6	12.6	
7	14.7	
8	16.8	
9	18.9	
Prop. Parts		

Hence, without regard to the decimal point, $m = 302 + \frac{6}{14} = 3024+$. Pointing off properly,

$$m = \text{anti-log } 0.4806 = 3.024+.$$

2. $\log m = 7.0959 - 10. \qquad m = ?$

$$\left.\begin{array}{r} \left.\begin{array}{rl} \text{mantissa of log } 124 = & .0934 \\ \text{mantissa of log } m = & .0959 \end{array}\right\} 25 \\ \text{mantissa of log } 125 = .0969 \qquad \end{array}\right\} 35$$

Hence m has the sequence of figures

$$124 + \tfrac{25}{35} = 1247+.$$

Pointing off properly,

$$m = \text{anti-log } (7.0959 - 10) = .001247+.$$

Note. The value of the quotient $\frac{25}{35}$ may be obtained from the column of Prop. Parts by finding the number of tenths of 35 required to equal 25. We have from this column,

$$.7 \times 35 = 24.5 \text{ and } .8 \times 35 = 28.0.$$

Hence we see that to make 25 we need a little more than $.7 \times 35$. A close approximation would be .71+, making $m = .0012471+$.

When the tabular difference is large, it is possible to obtain correctly more than four significant figures of a number when its four-place logarithm is given.

46. Exercises.

In the following exercises, the base is understood to be 10, and four-place logarithms are to be used.

1. Find log 831, log 8.31, log .831, and log .0831.

2. Find log 78.03, log .073, log .00284.

3. Find the numbers whose logarithms are 2.6133; 1.4720; 0.2211.

4. Find the numbers whose logarithms are 8.5320 − 10; 6.7019 − 10.

5. Find the approximate value of 564.1 × .0065.

6. Calculate $\sqrt[3]{154.2}$ and $(7.541)^3$.

7. Calculate 518 ÷ 313 and 25.03 ÷ 2.14.

8. Calculate .001022 ÷ .0000513 × 1.415.

9. Calculate $17\sqrt[3]{29}$ and $41\sqrt{0.512}$.

10. Calculate $\sqrt[5]{0.35^4} \times \sqrt[3]{0.47^2}$.

11. Calculate $\dfrac{(.00165)^3(.0764)^2}{(.00346)^4}$.

12. Calculate $\sqrt[3]{214 - \sqrt{214}}$.

13. Show that $\dfrac{1}{\sqrt[7]{7\sqrt[7]{7}}} = 0.9612$.

14. In each of the following equations, calculate by logarithms the value of the quantity not given:

(*a*) $n = \dfrac{k}{\sqrt{l}}; \qquad k = 6.35; \qquad l = 2.45.$

(*b*) $v = \sqrt{2gh}; \qquad g = 32.2; \qquad h = 2775.$

(c) $p = av^{-b}$; $a = .0018$; $v = 1000$; $b = 1.4$.

(d) $d = \sqrt{\frac{kl}{R}}$; $k = 9.65$; $l = 1320$; $R = 17.4$.

Write as a single term:

15. $\log a - \log b + \log c - \log d$.

16. $3 \log x - 4 \log y + 2 \log z$.

17. $\frac{1}{2} \log u + \frac{1}{3} \log v - \frac{1}{4} \log w$.

18. $\log \frac{a}{b} + \log \frac{b}{c} + \log \frac{c}{d} - \log \frac{ax}{dy}$.

19. $3 \log a - \log (x + y) - \frac{1}{3} \log (ax + b) + \log \sqrt{ax + b}$.

Use logarithms in the calculations required below.

20. What is the weight of water which will fill a rectangular dish whose dimensions are 12.5 × 8.6 × 3.2 inches, if water weighs 62.4 lb. per cu. ft.?

21. What is the diameter of a cylindrical pint can whose height is 6.25 inches? Take $\pi = 22/7$.

22. What is the surface of a cube whose volume is 828.5 cu. in.?

23. What is the weight of a cork ball 10 ft. in diameter, if the cork weighs 20 lb. per cu. ft.? Use the formula for the volume of a sphere, $V = \frac{4}{3} \pi r^3$, taking $\pi = 3.142$. (First make an off-hand guess at the weight.)

24. What is the radius of a sphere whose volume is 1000 c.c.? Take $\pi = 3.142$.

25. The volume of a right circular cone of height h and radius r is $V = \frac{1}{3} \pi r^2 h$. If $V = 15.25$ cu. in. and $h = 2.48$ in., find r. Take $\pi = 22/7$.

47. Compound Interest. — The calculation of compound interest furnishes a good example of the use of logarithms.

Suppose a principal, P dollars, to be placed at interest at the rate r, measured in fractions of a dollar, compounded annually. (A rate in per cent, as 5 per cent, would mean $r = .05$.)

The interest for one year is Pr dollars; adding this to the principal we have the amount at the end of the first year.

$$A_1 = P + Pr = P(1 + r).$$

That is, *to get the amount at the end of the year, multiply the principal by* $(1 + r)$.

During the second year the principal is A_1; hence its amount at the end of the second year,

$$A_2 = A_1 (1 + r) = P(1 + r)^2.$$

During the third year the principal is A_2; its amount at the end of the third year,

$$A_3 = A_2 (1 + r) = P(1 + r)^3.$$

Continuing, we find that the amount of P dollars placed at interest at the rate r compounded yearly for n years is

$$A_n = P(1 + r)^n.$$

If the interest is compounded every t years instead of annually, the amount *after n compoundings* is

$$A_n = P(1 + rt)^n.$$

Here t may be a fraction.

Thus if the rate is 6% and interest is compounded every month, we would have $t = \frac{1}{12}$ and $r = .06$.

Then $rt = .005$, and

$$\begin{aligned}
&\text{at the end of 1 month: } A_1 = P(1 + .005);\\
&\text{at the end of 2 months: } A_2 = A_1(1 + .005) = P(1 + .005)^2;\\
&\text{at the end of } n \text{ months: } A_n = P(1 + .005)^n.
\end{aligned}$$

This is the amount after n compoundings.

Example 1. Find the amount of $175 in 8 years, at 5 per cent, compounded annually.

$$\begin{aligned}
P &= 175; \quad r = .05; \quad n = 8.\\
A &= 175 \times (1 + .05)^8 = 175 \times (1.05)^8.\\
\log A &= \log 175 + 8 \log 1.05.\\
\log 1.05 &= 0.0212\\
8 \log 1.05 &= 0.1696\\
\log 175 &= \underline{2.2430}\\
\log A &= 2.4126; \qquad \mathbf{A = \$258.6.}
\end{aligned}$$

To get the result correct to the cent would require the use of a 6-place table of logarithms.

Example 2. Find the amount of $122 in 10 years at 6 per cent, interest compounded quarterly.

$$\begin{aligned}
P = 122; \quad rt &= .06 \times \tfrac{1}{4} = .015; \quad n = 10 \times 4 = 40.\\
A &= 122 \times (1.015)^{40}.\\
\log A &= \log 122 + 40 \log 1.015.\\
\log 1.015 &= 0.0064\tfrac{1}{2}\\
40 \log 1.015 &= 0.2580\\
\log 122 &= \underline{2.0864}\\
\log A &= 2.3444; \qquad \mathbf{A = \$221.0.}
\end{aligned}$$

48. Exercises. — Find the amounts in the following cases:

1. $1412 in 19 years at 4 per cent; compounded annually.

2. $1000 in 10 years at 4 per cent; compounded quarterly.

3. $1000 in 20 years at 4 per cent; compounded semi-annually.

4. $1565 in 14 years at 5 per cent; compounded biennially.

5. In 1626 Long Island was sold by the Indians for $24. If they had put the money in the bank at 7 per cent, compounded annually, what would be the amount at present?

The expansion may now be completed by the formula.

21. $(x + y - z)^3$. **22.** $(\sqrt{x} - \sqrt{y} + \sqrt{z})^3$.

23. $(1 - \sqrt{2} + \sqrt{-2})^4$. **24.** $(1 + \sqrt{3} - i\sqrt{2})^3$.

25. $(1 + 2x + 3x^2 + 4x^3)^2$.

Calculate:

26. the 7th term of $(3 + 2x^2)^9$.

27. the 4th term of $(\sqrt{2c} + \sqrt{3d})^{10}$.

28. the 6th term of $(2b^{\frac{2}{3}} - \frac{1}{2}\sqrt{x})^8$.

29. the 10th term of $(3y^{\frac{2}{3}} + \frac{1}{3}y^{\frac{3}{2}})^{15}$.

30. the 9th term of $(\sqrt{\frac{2}{3}a^3} - \sqrt{\frac{1}{2}a^5})^{20}$.

54. Approximate Computation by Use of the Binomial Theorem. — When x is a small fraction, the terms of the formula

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \cdots$$

rapidly decrease. *In any numerical problem in which only approximate results are required, retain only enough terms of the expansion to obtain the desired degree of accuracy.*

It will often be found sufficient to use the simple formula,

$$(1 + x)^n = 1 + nx, \text{ approximately.}$$

Example 1. Calculate $(0.997)^4$ to three decimals.

$$(0.997)^4 = (1 - .003)^4 = 1 - 4 \times .003 = 0.988.$$

Exercise. Show that the terms neglected will not affect the third decimal place.

Example 2. Calculate $(2.05)^3$ to three decimals.

$$\begin{aligned}(2.05)^3 = 2^3(1 + .025)^3 &= 8(1 + 3 \times .025 + 3 \times .000625 + \cdots) \\ &= 8 \times 1.0769 = 8.615.\end{aligned}$$

Exercises. Calculate to three decimal places the value of:

1. $(0.995)^5$. **2.** $(1.05)^7$. **3.** $(3\frac{1}{15})^4$.

4. $(2\frac{1}{16})^4$. **5.** $(3.998)^6$. **6.** $(8.0125)^2$.

7. Calculate the value of $(.99995)^7$ to seven decimals.

CHAPTER IV

LINEAR EQUATIONS

55. Axioms Used in Solving Equations. — If

$$X = Y, \quad \text{and} \quad m = n,$$

then

$$X + m = Y + n, \quad X - m = Y - n,$$

$$mX = nY, \quad \text{and} \quad \frac{X}{m} = \frac{Y}{n}.$$

That is, *if both members of an equation be increased or diminished, multiplied or divided, by the same or equal quantities, the results are equal.*

Also if $X = Y$, then $X^n = Y^n$, and $\sqrt[n]{X} = \sqrt[n]{Y}$, provided the corresponding nth roots of X and Y are selected.

That is, *like powers or like roots of equal quantities are equal.*

If

$$X + m = Y,$$

then subtracting m from both members,

$$X = Y - m.$$

That is, *a term may be transposed from one side of an equation to the other provided its sign is changed at the same time.*

When the members of an equation involve sums or differences of fractions, *the equation may be cleared of fractions by multiplying both members by the L. C. D. of the several fractions.*

56. Linear Equation. — If x be an unknown quantity related to the known quantities a and b through the equality $ax + b = 0$, this equation being called *the standard form of the linear equation in one unknown,* we obtain the value of x as

$$x = -\frac{b}{a}.$$

Every linear equation in one unknown may be solved by reducing it to standard form and applying the last formula.

The reduction of an equation to standard form will involve some or all of the following steps:

1. Clearing of radicals. (**39**, after exercise 137.)
2. Clearing of fractions.
3. Expanding products or powers of polynomials.
4. Transposing and cancelling.
5. Collecting terms.

To verify the value found, substitute it in the given equation. The result should be an identity.

Example 1. Solve for x: $(1 + b)\,x + ab = b\,(a + x) + a.$
Expanding the products: $x + bx + ab = ab + bx + a.$
Cancelling like terms: $x = a.$
Check: $(1 + b)\,a + ab = b\,(a + a) + a.$

Example 2. $$\frac{1}{2} + \frac{2}{x + 2} = \frac{x + 2}{2\,x}.$$

Multiplying by the *L. C. D.*, $2\,x\,(x + 2)$:

$$x\,(x + 2) + 4\,x = (x + 2)^2.$$

Expanding: $x^2 + 2\,x + 4\,x = x^2 + 4\,x + 4.$
Cancelling: $2\,x = 4$ or $x = 2.$
Check: $\frac{1}{2} + \frac{2}{4} = \frac{4}{4}.$

Example 3. Solve for x: $\sqrt{x + 20} - \sqrt{x - 1} - 3 = 0.$

Transposing: $\sqrt{x + 20} = \sqrt{x - 1} + 3.$
Squaring: $x + 20 = x - 1 + 6\sqrt{x - 1} + 9,$
or $2 = \sqrt{x - 1}.$
Squaring: $4 = x - 1$ or $x = 5.$
Check: $\sqrt{25} - \sqrt{4} - 3 = 0.$

57. Infinite Solutions. — Consider the equation $\frac{1}{x + 1} = \frac{1}{x - 1}\cdot$ Since $x + 1$ cannot equal $x - 1$ for any value of x, there is no value of x which will satisfy the given equation.

But if we substitute in the given equation successively $x = 10, 100, 1000,$ etc., the equation is more nearly satisfied, the larger the value of x. We can take x so large as to make the difference between the two members of the equation as small as we please; for this difference is

$$\frac{1}{x + 1} - \frac{1}{x - 1} = \frac{-2}{x^2 - 1}.$$

For brevity we say that $x = \infty$ *is a solution of the equation*, meaning thereby that as increasing values of x are substituted, the equation is more and more nearly satisfied.

Substituting formally $x = \infty$, we obtain

$$\frac{1}{\infty + 1} = \frac{1}{\infty - 1} \quad \text{or} \quad 0 = 0.$$

The equation of example 2 of (56) admits the solution $x = \infty$. This will be evident on putting ∞ for x in

$$\frac{1}{2} + \frac{2}{x+2} = \frac{x+2}{2x} = \frac{1}{2} + \frac{1}{x}.$$

58. Exercises.—Solve for x, including infinite solutions when present:

1. $5(a - x) = 3(b - x)$.
2. $p(x - 1) + x = q - p$.
3. $a(bx - c) = ac - abx$.
4. $\frac{m - x}{n} = \frac{x - n}{m}$.
5. $\frac{m}{x} + n = \frac{p}{x} + q$.
6. $\frac{a + bx}{c + dx} = \frac{a}{c}$.
7. $\frac{a + bx}{c + dx} = \frac{b}{d}$.
8. $\frac{a + b}{a + bx} = \frac{c - d}{c - dx}$.
9. $\frac{\frac{a+1}{x}}{\frac{b}{x}} = \frac{a + x}{b + x}$.
10. $\frac{\frac{cx + d}{m}}{\frac{cx}{d}} = \frac{2d}{m}$.
11. $\frac{m}{x - m} - \frac{n}{x - n} = \frac{m - n}{x - a}$.
12. $\sqrt{x + 15} + \sqrt{x - 13} = 14$.
13. $3\sqrt{16x + 9} + 9 = 12\sqrt{4x}$.
14. $\sqrt{x} - \sqrt{x - 5} - \sqrt{5} = 0$.
15. $\sqrt{\frac{15}{4} + x} = \frac{3}{2} + \sqrt{x}$.
16. $\frac{1}{x} + \frac{2}{x} + \frac{3}{x} = 0$.

Problems.

1. If 12 be added to 7 times a certain number the sum is 54. What is the number?

2. Find a number such that if 16 be subtracted from it and the result multiplied by 5, the product equals the number.

3. Find a number such that if a be subtracted from it and the result multiplied by m, the product equals the number.

4. Find a number such that 3 times the number increased by 10 equals 5 times the number.

5. Find a number such that m times the number increased by a equals n times the number.

6. The age of a boy is three times that of his brother, and their combined ages make 16 years. How old is each?

7. In what proportion must two liquids, of specific gravities 1.20 and 1.40 respectively, be mixed to form a liquid of specific gravity 1.25?

8. Two boys start together and walk around a circular half-mile track at the rates of 3½ and 4 miles an hour respectively. After how many laps will they pass each other?

9. A can do a piece of work in 3 days, B in 5 days. How long will it take them both to do it?

10. A can do a piece of work in a days, B in b days. How long will it take them both to do it?

11. A can do a piece of work in a days, B in b days, and C in c days. In how many days can they together do it?

12. At what time between 4 and 5 are the hands of a clock together?

13. At what time between 10 and 11 are the hands of a clock at right angles? Opposite each other?

14. The sum of the ages of A, B, and C is 60 years. In how many years will the sum be 5 times as great as it was 10 years ago?

15. Water flows into a cistern through two pipes A and B, and out through a third pipe C. The cistern can be filled by A in 1 hour, by B in 45 minutes, and emptied by C in 36 minutes. How long will it take to fill the empty cistern when all three pipes are running?

59. Graphic Solution of Linear Equations. — Suppose that a given equation has been reduced to the standard form,

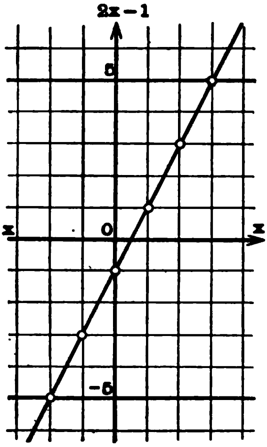
$$ax + b = 0.$$

The solution of the equation is that value of x which reduces $ax + b$ to zero. We shall now show how to indicate this value of x by a diagram.

Example. Graphic solution of the equation $2x - 1 = 0$.

Make a table showing the values of the expression $2x - 1$ for a series of assumed values of x, thus:

x.	$2x - 1$.
-2	-5
-1	-3
0	-1
$+1$	$+1$
$+2$	$+3$
$+3$	$+5$
.	.
.	.



Draw a straight line xx, and on it mark a number scale (**31**) to indicate the values of x. Cross-ruled paper should be used. At $x = 0$ on line xx draw a line perpendicular to it, and on this new line mark off a number scale to be used for measuring the values of $2x - 1$, upward if positive, downward if negative. Usually the two scales are marked off with the same unit of length, but this is not necessary. It is often convenient to use different scales on the two lines.

Now start from the point where $x = -2$ on the x-scale, and lay off a perpendicular 5 units long and *downward*, to indicate that, when $x = -2$, $2x - 1 = -5$. At $x = -1$ draw a perpendicular 3 units long and downward, at $x = 0$ a perpendicular 1 unit long and downward. At $x = 1, 2, 3$ respectively draw perpendiculars 1, 3, 5 units long and *upward*, since $2x - 1$ is now positive.

The free ends of these perpendiculars will be found to lie on a straight line. Draw this line. If we choose any other value of x and draw the perpendicular to represent the corresponding value of $2x - 1$, its free end will be found to fall on this straight line. Hence we call this line the **graphic representation**, or **graph**, of the expression $2x - 1$.

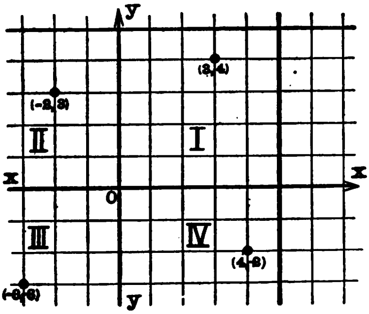
The point where the graph of the expression $2x - 1$ crosses the x-scale, or **x-axis**, marks the value of x for which $2x - 1$ equals zero; hence this is the solution of the equation $2x - 1 = 0$.

Exercises. Obtain graphic solutions of the following equations:

1. $2x - 7 = 0$. **2.** $\frac{2}{3}x - 1 = 0$. **3.** $3x + 5 = 0$. **4.** $3x + 7 = 0$.

60. Coördinates. — In diagrams like that above we find it convenient to use a single letter, as y, to represent the given expression, as $2x - 1$. In general we use y to denote lengths of perpendiculars to the x-axis; the scale on the vertical line, or **y-axis**, is used to measure the values of y.

By constructing suitable values of x and y, we can reach any point in the plane of the lines Ox and Oy. Such a point is indicated by the symbol (x, y), and x and y are called the **coördinates** of the point, x being named the **abscissa**, and y the **ordinate.** Let the student give the values of the ordinate and abscissa of the points marked in the figure and of several other selected points.



The four quarters into which the plane is divided by the lines Ox and Oy are called quadrants, and are numbered as shown. Let the student name the quadrant of each of the points (4, 1), (2, −3), (−5, −4), (−1, 3); also make a diagram showing these points.

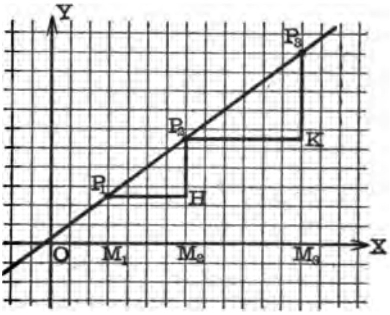
61. Graph of $ax + b$ is a Straight Line. — This is the general form of which the expression $2x - 1$ is a special case. Let y represent it, so that

$$y = ax + b.$$

To draw the graph of the expression $ax + b$, where a and b are any given real numbers, we proceed exactly as for $2x - 1$ in (**59**); that is, we calculate the values of $ax + b$ for a series of chosen values of x and construct perpendiculars of the proper lengths. We shall now show that the ends of all such perpendiculars must lie on a straight line, by showing that the ends of any three selected at random do so.

Let x_1, x_2, and x_3 be any three values of x; let y_1, y_2, and y_3 be the corresponding values of y. Lay off the pairs of values (x_1, y_1), (x_2, y_2), and (x_3, y_3) so that (see figure)

$$\begin{array}{ll} x_1 = OM_1, & y_1 = M_1P_1, \\ x_2 = OM_2, & y_2 = M_2P_2, \\ x_3 = OM_3, & y_3 = M_3P_3. \end{array}$$



But since y_1 is the value of y obtained by putting $x = x_1$ in $y = ax + b$, and similarly for y_2 and y_3, we have

$$\begin{array}{l} y_1 = ax_1 + b, \\ y_2 = ax_2 + b, \\ y_3 = ax_3 + b. \end{array} \quad \therefore \quad \begin{array}{l} y_2 - y_1 = a(x_2 - x_1), \\ y_3 - y_2 = a(x_3 - x_2). \end{array}$$

Therefore, $$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} \; (= a)$$

But $$\begin{aligned} y_2 - y_1 &= M_2P_2 - M_1P_1 = M_2P_2 - M_2H = HP_2; \\ y_3 - y_2 &= M_3P_3 - M_2P_2 = KP_3; \\ x_2 - x_1 &= OM_2 - OM_1 = M_1M_2 = P_1H; \end{aligned}$$

and $$x_3 - x_2 = OM_3 - OM_2 = M_2M_3 = P_2K.$$

Substituting these in the two fractions above we obtain

$$\frac{HP_2}{P_1H} = \frac{KP_3}{P_2K}.$$

Therefore $\triangle\ P_1HP_2$ is similar to $\triangle\ P_2KP_3$ and $\angle\ HP_1P_2 = \angle KP_2P_3$. Hence the points P_1, P_2, P_3 lie on a straight line.

Theorem: The graph of the equation $y = ax + b$ *is a straight line.*

Corollary: To construct the graph of the equation $y = ax + b$, *construct two points on it and draw a straight line through them.*

62. Exercises.—Draw the graphs of the equations (each set to the same reference lines):

1. $y = x + 1$, $2y = 2x + 2$, $5y = 5x + 5$, $\frac{1}{2}y = \frac{1}{2}x + \frac{1}{2}$.
2. $y = 3x - 4$, $2y = 6x - 8$, $ky = 3kx - 4k$.
3. $y = x + 1$, $y = x + 2$, $y = x + 3$, $y = x - 1$.
4. $y = 3x - 4$, $y = 3x - 2$, $y = 3x$, $y = 3x + 1$.
5. $y = x + 1$, $y = 2x + 1$, $y = 3x + 1$, $y = \frac{1}{2}x + 1$.
6. $y = 3x - 4$, $y = 6x - 4$, $y = 9x - 4$, $y = \frac{3}{2}x - 4$.
7. $y = -x + 1$, $y = -3x - 4$.
8. $y = x - 1$, $y = 3x + 4$.

Explain the effect on the graph of $y = ax + b$, of:

9. Multiplying the equation through by a constant.
10. Changing the value of b.
11. Changing the value of a.
12. Changing the sign of b.
13. Changing the sign of a.

63. Linear Equation in Two Variables.—Such an equation has the general form

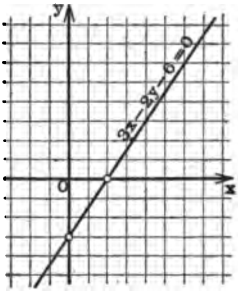
$$Ax + By + C = 0. \tag{1}$$

If we consider the variables x and y as the coördinates of a moving point, then all possible pairs of values of x and y which satisfy equation (1) will yield a series of points unlimited in number; this entire collection of points is called the **graph** of the equation.

Theorem: The graph of the linear equation $Ax + By + C = 0$ *is a straight line.*

Proof: If $B \neq 0$, we can write $y = -\frac{A}{B}x - \frac{C}{B}$, which has the form $y = ax + b$. Therefore the graph is a straight line when $B \neq 0$.

If $B = 0$, the equation reduces to $Ax + C = 0$, or $x = -\frac{C}{A}$, unless $A = 0$. But if $A = 0$ and $B = 0$, then $C = 0$ and the equation vanishes identically. Excluding this, we may reduce $Ax + C = 0$ to $x = -\frac{C}{A}$, or $x =$ a constant. But this is a straight line parallel to the y-axis. Therefore the given linear equation represents a straight line. (Hence the term "linear" equation.)



Example. $3x - 2y - 6 = 0.$

Since the equation represents a straight line, we need to locate only two points; putting $x = 0$ gives $y = -3$, hence one point is $(0, -3)$; putting $y = 0$ gives $x = 2$, hence another point is $(2, 0)$. The graph is shown in the figure.

64. Exercises. — Draw graphs for the following equations:

1. $3x + 2y - 6 = 0.$ **2.** $x - 3y + 6 = 0.$
3. $3x + 4y = 12.$ **4.** $6x - 2y + 12 = 0.$
5. $x + y = 1.$ **6.** $2x - 3y = 7.$

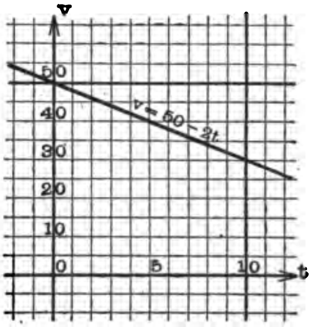
7. Show that the equations $Ax + By + C = 0$ and $y = -\frac{A}{B}x - \frac{C}{B}$ have the same graph.

8. Show that the equations $Ax + By + C = 0$ and $kAx + kBy + kC = 0$ have the same graph, k being any constant.

9. How is the graph of $Ax + By + C = 0$ affected by a change in C? in B? in A?

10. In the figure of (63), showing the graph of $3x - 2y - 6 = 0$, scale off the values of x and y for any point selected at random on the line; do these values satisfy the equation? Should they do so? Why?

65. Use of the Graph. — When any two variable quantities are connected by a *linear equation*, the relation between them can always be represented graphically by a straight line. It is only necessary to consider the two quantities as the coördinates of a point.



Example 1. A vessel containing 50 gallons of water is being emptied at the rate of 2 gallons per second. How much water is left at the end of t seconds?

Solution. Let v be the volume of water left at the end of t seconds. Since $2t$ gallons have run out, the amount remaining is

$$v = 50 - 2t.$$

This is a linear equation between the two variables t and v, and may be represented graphically. We shall consider that t takes the place of x, and v of y. The graph is shown in the preceding figure, one square on the t-scale representing one second, and one square on the v-scale representing five gallons.

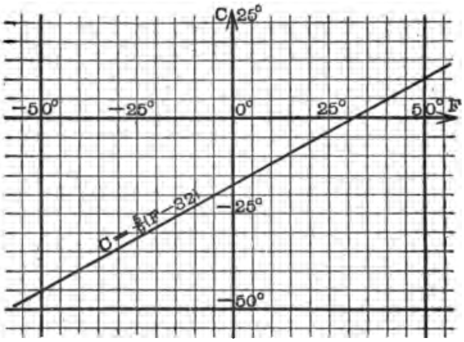
Exercise. By inspection of the graph determine how much water remains at the end of 0, 5, 8, 12 seconds respectively. Check by calculation. How far to the right may the diagram be continued?

Example 2. The relation between the readings on the scales of a Centigrade and a Fahrenheit thermometer is given by the equation

$$C = \frac{5}{9}(F - 32).$$

Draw the graph.

We shall retain the letters F and C instead of replacing them by x and y. The graph is shown in the adjacent figure. From it the reading of either scale corresponding to a given reading of the other may be at once read off, with an accuracy of about 1°.



Exercise. Read off the values of C corresponding to $F = -40°$, $F = 0°$, $F = 57°$ respectively; also the values of F when $C = -30°, 0°, +21°$.

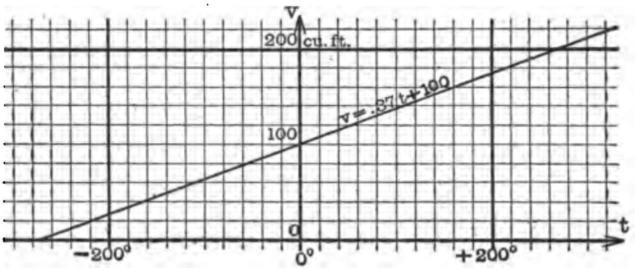
Example 3. A volume of gas expands when the temperature rises and contracts when the temperature falls according to the law

$$v = v_0(1 + \tfrac{1}{273}t),$$

where v_0 = volume at temperature 0°,

and v = volume at temperature t°.

Represent graphically the relation between volume and temperature for a quantity of gas whose volume at 0° is 100 cu. ft.



Replacing $\frac{1}{273}$ by its approximate value .0037, and v_0 by 100, the equation becomes

$$v = .37\,t + 100.$$

The graph is given in the figure.

66. Exercises.

1. From the figure, read off the volumes corresponding to the temperatures 250°, 75°, 0°, and $-273°$; also the temperature corresponding to the volumes 150, 75, and 20 cu. ft. respectively.

2. Construct a graphic conversion table for converting yards to feet.

3. Construct a graph showing the relation between the circumference and the diameter of a circle. (See p. 105.)

4. A falling body, starting with an initial velocity of v_0 ft. per second, acquires in t seconds a velocity given by $v = gt + v_0$, in which $g = 32.2$. Assume a value of v_0 and draw the graph of the equation.

5. Let A be the lateral area of a right circular cylinder of height h and radius of base r. Draw the graph showing the relation between A and h when r is fixed. Also draw the graph showing the relation between A and r when h is fixed. Lateral area of cone $= 2\pi rh$.

6. Same as 5, except that cone is substituted for cylinder, and slant height for height. Lateral area of cone $= \pi rh$.

67. Simultaneous Linear Equations. — Let there be given two linear equations containing two unknown quantities x and y, as

$$ax + by + c = 0,$$
$$a'x + b'y + c' = 0.$$

It is required to obtain all pairs of values of x and y which satisfy both equations simultaneously.

First Method — By Substitution. — Solve one of the equations for either of the unknowns in terms of the other; substitute the value so found in the second equation, thus obtaining a linear equation in one unknown; solve for this unknown and then obtain the other by substitution in either of the given equations.

Check. Substitute the values of x and y in the equation not used in the last step of the solution.

Example. Solve for x and y:

$$\frac{x+y}{3} + x = 15 \quad \text{and} \quad \frac{x-y}{5} + y = 6.$$

Clearing and simplifying:

$$4x + y = 45 \quad \text{and} \quad x + 4y = 30.$$

From the first of these, $y = 45 - 4x.$

Substituting in the second, $x + 4(45 - 4x) = 30.$

Hence $15x = 150$ or $x = 10.$

Then $y = 45 - 4x = 5.$

Check. $$\frac{x-y}{5} + y = \frac{10-5}{5} + 5 = 1 + 5 = 6.$$

Second Method — By Multipliers. — Multiply the first equation by a', the second by $-a$, and add the resulting equations together. This eliminates x, and yields a linear equation in y alone, from which y may be found. Similarly x is found by multiplying the first equation by b', the second by $-b$, and adding. The proper multipliers for the two eliminations are conveniently indicated thus:

$$\begin{array}{|r|r|l} b' & a' & ax + by + c = 0, \\ -b & -a & a'x + b'y + c' = 0. \\ \hline \end{array}$$

Check. Substitute the values of x and y in either of the given equations.

Example. Solve for x and y:

$$8x - 15y + 30 = 0 \text{ and } 2x + 3y - 15 = 0.$$

Indicating the multipliers:

$$\begin{array}{|r|r|l} 3 & 2 & 8x - 15y + 30 = 0 \\ 15 & -8 & 2x + 3y - 15 = 0 \\ \hline \end{array}$$

$$\begin{aligned} -54y + 180 &= 0, \quad \text{or} \quad y = \tfrac{10}{3}, \\ 54x - 135 &= 0, \quad \text{or} \quad x = \tfrac{5}{2}. \end{aligned}$$

Check. $8 \times \frac{5}{2} - 15 \times \frac{10}{3} + 30 = 20 - 50 + 30 = 0.$

Third Method — By Determinants. — See (**161**).

68. Exceptional Cases.

1. *The given equations are not independent.*

In this case one equation is a multiple of the other, so that

$$a = ka', \; b = kb', \text{ and } c = kc',$$

k being a constant. Both equations are then equivalent to a single one, and do not suffice to determine two unknowns.

By assuming any value for x, substituting in one of the equations and solving for y, we obtain a pair of values which satisfy *both* equations. (Why?) Hence there exists an infinite number of solutions.

2. *The given equations are inconsistent.*

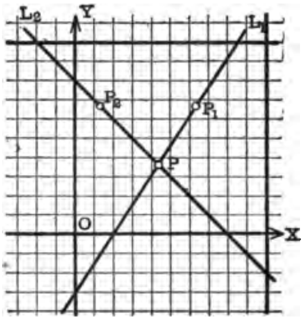
If $a = ka'$ and $b = kb'$, but $c \neq kc'$, then the given equations are self-contradictory. For if we subtract k times the second equation from the first, we obtain $c = kc'$, which is not true.

In this case there is no finite solution possible. For if we assume $x = x_1$ and $y = y_1$ to be a solution of either equation, the other equation will not be satisfied by these values because $c \neq kc'$.

69. Graphic Solution of Two Simultaneous Linear Equations. Let the equations be

(1) $ax + by + c = 0,$

(2) $a'x + b'y + c' = 0.$



The graph of each equation is a straight line. Suppose L_1 and L_2 (figure) to be the graphs of equations (1) and (2) respectively. Then the coördinates of any point on L_1, as P_1, satisfy equation (1) and of any point P_2 on L_2 satisfy (2). Hence *the coördinates of the intersection P of L_1 and L_2 satisfy both equations simultaneously* and give the required solution.

Exceptional Cases.

1. *The given equations are not independent.*

Then, as before, $a = ka'$, $b = kb'$, and $c = kc'$. The lines L_1 and L_2 will coincide and have an infinite number of common points.

2. *The given equations are inconsistent.*

Then $a = ka'$, $b = kb'$, but $c \neq kc'$. The lines L_1 and L_2 are now parallel to each other, but not coincident. Hence they have no common point (except at infinity). Including the infinite solution is equivalent to the statement "parallel lines meet at infinity."

70. Exercises.—Solve for x and y, including graphical solutions:

1. $2x + y = 11.$
$3x - y = 4.$
2. $3x + 8y = 19.$
$3x - y = 6.$
3. $4x + 3y = 8.$
$2x - y = 14.$
4. $3x - 4y = 85.$
$5x - 4y = 107.$
5. $5x - 7y = 101.$
$7x + y = 55.$
6. $2x - y - 1 = 0.$
$8x - 4y - 3 = 0.$
7. $15x - 7y = 9.$
$9y - 7x = 13.$
8. $2x - 7y = 8.$
$4y - 9x = 19.$
9. $x - 2y + 2 = 0.$
$4x - 8y + 2 = 0.$
10. $8x + 3y = 25.$
$12x - 9y = 70.$
11. $\frac{2}{3}x - \frac{1}{2}y - 2 = 0.$
$x - \frac{3}{4}y - 3 = 0.$
12. $3y + 4x - 1 = 0.$
$18 + 3x = 4y.$
13. $2x = 11 + 3y.$
$3x - 15 = 4y.$
14. $4x + 7y = 52.$
$6x - 5y = 16.$
15. $3x + 4y - 5 = 0.$
$\frac{1}{2}x + \frac{2}{3}y - \frac{5}{6} = 0.$
16. $10y - 2x = 21.$
$13x - 8y = 120.$

17. $\frac{x}{2} - \frac{y}{3} = 7.$

$2x - 3y = 48.$

18. $\frac{7x}{6} + \frac{10y}{3} = 34.$

$\frac{7x}{8} + \frac{y}{4} = 12.$

19. $4 + y = \frac{9x}{4}.$

$3x - 8 = \frac{4y}{5}.$

20. $\frac{2x}{3} = 10 - \frac{3y}{2}.$

$\frac{57y}{4} = 5x - 7.$

SIMULTANEOUS LINEAR EQUATIONS IN MORE THAN TWO UNKNOWNS

71. Three Equations in Three Unknowns.

Let the given equations be,

$$ax + by + cz + d = 0, \tag{1}$$
$$a'x + b'y + c'z + d' = 0, \tag{2}$$
$$a''x + b''y + c''z + d'' = 0. \tag{3}$$

Eliminate one of the variables, say z, from two pairs of the equations, as from (1) and (2) and from (2) and (3). Solve the resulting equations for x and y. Substitute the values of x and y so found in one of the given equations and solve the result for z. See also (**162**).

Check. Substitute the values of x, y, and z so found in either of the equations not used in the last step of the solution.

Exceptional Cases.

1. *The given equations are not independent.*

(a) In this case one of the equations can be expressed as a linear combination of the other two, with constant coefficients. Hence any solution of these two equations is also a solution of the third. But two equations in three variables admit *an infinity of solutions.* For we can choose any value for z at pleasure, substitute it in the two equations and obtain a pair of values of x and y.

(b) It may happen that two of the equations can be expressed as multiples of the third. Then any solution of the third equation is also a solution of the other two. Hence again there exists *an infinity of solutions,* since we may choose for two of the variables any value at pleasure and obtain the corresponding value for the third.

2. *The equations are inconsistent.*

In this case the equations in x and y obtained by eliminating z are also inconsistent. Hence there is *no solution* (except the infinite solution).

We shall not discuss here the graphic solution of three linear equations in three variables. Interpreted graphically, each of the equations (1), (2), and (3) represents a plane in space. In general, three planes meet in a single point, giving one and only one solution. The exceptional cases are:

1. (a) The three planes meet in a common line. Hence any point in this line gives a solution.

(b) The three planes coincide. Hence any point in one of the planes is a solution.

2. The three planes are parallel. No solution, except infinity. ("Parallel planes meet at infinity.")

72. Four Equations in Four Unknowns.—*Solution.* Eliminate one of the unknowns from three different pairs of the four given equations. The three resulting equations can be solved for the other three unknowns. The fourth unknown is then found by substituting these three in one of the given equations.

Check. Substitute the values of the four unknowns in one of the equations not used in the last step of the solution.

Exceptional cases arise, quite analogous to the preceding. We shall not discuss them here.

The method of solution outlined above is evidently applicable to any number of linear equations in the same number of variables. A more convenient method involves the use of determinants. (Chapter XI.)

73. Exercises and Problems.

1. $\frac{x}{4}+\frac{y}{3}=3\frac{1}{2}.$
$\frac{x}{3}-\frac{y}{4}=\frac{1}{2}.$

2. $\frac{x}{3}+\frac{y}{2}=\frac{4}{3}.$
$\frac{x}{2}+\frac{y}{3}=\frac{7}{6}.$

3. $\frac{x+y}{3}+\frac{y-x}{2}=9.$
$\frac{x}{2}+\frac{x+y}{9}=5.$

4. $\frac{x+y}{4}+\frac{x-y}{6}=5.$
$\frac{x+y}{2}-\frac{x-y}{3}=10.$

5. $\frac{x-1}{8}+\frac{y-2}{5}=2.$
$2x+\frac{2y-5}{3}=21.$

6. $\frac{4x-5y}{40}=x+y.$
$\frac{2x+y}{3}-2y=\frac{1}{2}.$

7. $.25x+6y=10.$
$4.5x-8y=6.$

8. $4.2y+8x=33.$
$0.77y-0.6x=2.95.$

9. $0.2525x+0.03y=280.$
$3.122x+0.005y=3096.$

10. $0.2y+0.25x=2(y-x).$
$0.8x-3.7y=-15.3.$

11. $0.1y + 0.3x = 0.3.$
$0.05y + 0.15x = 0.15.$

12. $\frac{3}{4}x - 1.2y = 0.$
$\frac{5(x-1)}{2(4y+3)} = \frac{5}{6}.$

13. $\frac{1}{x} - \frac{1}{y} = \frac{11}{30}.$
$\frac{1}{x} + \frac{1}{y} = \frac{1}{30}.$

14. $\frac{3}{x} + \frac{4}{y} = 3.$
$\frac{15}{x} - \frac{2}{y} = 4.$

15. $\frac{1}{x} + \frac{6}{y} = 10.$
$\frac{4}{x} + \frac{9}{y} = 20.$

16. $\frac{2(5-11x)}{11(x-1)} + \frac{11-14y}{3-2y} = 5.$
$\frac{7+2x}{3-x} - \frac{125-288y}{36(2y+5)} = 2.$

17. $\frac{7-12x}{10y-19} = \frac{4-6x}{5y-11}.$
$\frac{12x-10y-17}{6x-5y+2} = \frac{8x-14y-5}{4x-7y+12}.$

18. $\frac{1}{1-x-y} - \frac{1}{x-y-1} = \frac{2}{3}.$
$\frac{1}{\frac{1}{1-x-y} - \frac{1}{1-x+y}} = \frac{3}{4}.$

19. $\frac{1}{x + \frac{1}{3y - \frac{5}{x}}} = \frac{1}{x - \frac{1}{3y - \frac{7}{x}}}.$
$\frac{1}{3y}\left(1 - \frac{1}{x}\right) = 1.$

20. $ax - by = n.$
$cx + dy = m.$

21. $x + 2y = 3a - 2b.$
$x - 2y = 2a - 3b.$

22. $\frac{x}{a} + \frac{y}{b} = c.$
$\frac{x}{d} = y.$

23. $\frac{x}{a} - \frac{y}{b} = c.$
$\frac{x}{g} - \frac{y}{f} = 0.$

24. $\frac{x}{m} + \frac{y}{n} = p.$
$\frac{x}{r} - \frac{y}{s} = q.$

25. $\frac{mx}{n} - \frac{py}{q} = t.$
$\frac{x}{n} + \frac{ry}{s} = v.$

26. $\frac{x+y-1}{x-y+1} = m.$
$\frac{y-x+1}{x-y+1} = mn.$

27. $\frac{x-m}{y} = \frac{p}{q}.$
$\frac{m}{x} = \frac{n}{y}.$

28. $\frac{a}{b+y} = \frac{b}{a-x}.$
$\frac{c}{d-x} = \frac{d}{c+y}.$

29. $\sqrt{x+1} - \sqrt{y-1} = 4.$
$\sqrt{x+1} + \sqrt{y-1} = 7.$

30. $2\sqrt{x-y} + y + 1 = 0.$
$\sqrt{x-y} + 2y - 2 = 0.$

31. $\frac{1}{\sqrt{x-y}} + \frac{2}{\sqrt{x+y}} + 3 = 0.$
$\frac{3}{\sqrt{x-y}} - \frac{2}{\sqrt{x+y}} + 1 = 0.$

32. $\frac{2}{ax+by} + \frac{x}{y} = 5.$
$\frac{3x}{y} - \frac{1}{ax+by} + 2 = 0.$

33. $x + y = 37.$
$x + z = 25.$
$y + z = 22.$

34. $x + y = a.$
$x + z = b.$
$y + z = c.$

35. $2x + 3y = 12.$
$3x + 2z = 11.$
$3y + 4z = 10.$

36. $\frac{1}{5}x - \frac{1}{2}y = 0.$
$\frac{1}{3}x - \frac{1}{2}z = 1.$
$\frac{1}{2}z - \frac{1}{3}y = 2.$

37. $1\frac{1}{3}x + 1\frac{1}{2}y = 10.$
$2\frac{2}{3}x + 2\frac{2}{5}z = 20.$
$3\frac{1}{4}y + 3\frac{2}{5}z = 30.$

38. $x + 2y + 3z = 32.$
$2x + 3y + z = 42.$
$3x + y + 2z = 40.$

39. $\frac{1}{y} + \frac{1}{z} = 2.$
$\frac{1}{x} + \frac{1}{z} = 4.$
$\frac{1}{x} + \frac{1}{y} = 6.$

40. $\frac{xy}{x+y} = \frac{1}{5}.$
$\frac{xz}{x+z} = \frac{1}{6}.$
$\frac{yz}{y+z} = \frac{1}{7}.$

41. $x + 2y = 5.$
$y + 2z = 8.$
$z + 2u = 11.$
$u + 2x = 6.$

42. $y + z + u = 2.$
$z + u + x = 3.$
$u + x + y = 4.$
$x + y + z = 5.$

43. $3x + y + z = 4.$
$x + 4y + 3u = 6.$
$6x + z + 3u = 8.$
$8y + 3z + 5u = 10.$

44. Find two numbers whose sum is 1293 and difference 381.

45. If at a given place the longest day exceeds the shortest night by 8 hours 10 minutes, what is the duration of each?

46. The sum of two numbers is 1000. Twice the first plus three times the second equals 2029. Find the numbers.

47. The annual interest on a capital is $180; at a rate of interest 1½% higher, the annual interest would be $240; find the capital and rate of interest.

48. A farmer sells 200 bushels of wheat and 60 bushels of corn for $345; 60 bushels of wheat and 200 bushels of corn would bring, at the same price per bushel, $240; find the price per bushel of each.

49. Two points move on the perimeter of a circle 740 ft. long; the one point, moving three times as fast as the second, overtakes it every 37 seconds; find the speed of each.

50. A vat of capacity 450 cu. ft. can be filled by two pipes. If the first pipe flows 3 minutes and the second 1 minute, 40 cu. ft. are discharged; if the first pipe flows 1 minute and the second 7 minutes, 60 cu. ft. are discharged. How long will it take both pipes to fill the tank, and what is the discharge per minute of each pipe?

51. How many pounds of copper, and how many of zinc, are contained in 186 pounds of brass (alloy of copper and zinc), if, when placed in water, 89 lbs. of copper lose 10 lbs. in weight, 7 lbs. zinc lose 1 lb., and 124 lbs. brass lose 15 lbs.?

52. An alloy of gold and silver weighing 40 lbs. loses 2½ lbs. when placed in water. How much gold and how much silver does it contain, if gold, when placed in water, loses $\frac{1}{19}$ of its weight, and silver $\frac{1}{10}$ of its weight?

53. Find the lengths of the sides of a triangle if the sum of the first and second is 30, of the first and third 33 and of the second and third 37.

54. Find three numbers which are in the ratio of 3 : 4 : 5 and whose sum is 480.

55. The contents of three measures are as 4 : 7 : 6; 10 measures of the first kind, 4 of the second, and 2 of the third together contain 20 gallons. How much does each measure contain?

56. A vessel may be filled by each of three measures as follows: by 4 of the first and 4 of the third, or by 20 of the first and 20 of the second, or by 28 of the first and 3 of the third. Also, the three measures together contain 29 pints. Find the content of each measure.

57. A vessel can be filled by three pipes: by the first and second in 72 minutes, by the second and third in 2 hrs., and by the first and third in 1½ hrs. How long will it take each pipe alone to fill the vessel?

58. A and B can do a piece of work in 12 days, B and C in 20 days, A and C in 15 days. How long will it take A, B, and C, working together, to do the job?

59. Three principals are placed at interest for a year, A at 4%, B at 5%, C at 6%; the interest on A and B is $796, on B and C $883, and on A and C $819. Find the amount of each principal.

60. Two bodies move on the circumference of a circle; when going in the same direction they meet every 30 seconds, and when going in opposite directions every 10 seconds; in the second case, when they are 30 ft. apart, they will again be 30 feet apart after 3 seconds. Find the speed of each body and the radius of the circle.

CHAPTER V

QUADRATIC EQUATIONS — EXPONENTIAL EQUATIONS

74. Quadratic Equation in One Variable. — Suppose we wish to find two numbers whose sum is 5 and whose product is 6.

Let $x =$ one of the numbers;

then $5 - x =$ the other number,

and $x(5 - x) = 6$

or $x^2 - 5x + 6 = 0.$

To determine x we must solve this equation.

Definition. An equation of the form

$$ax^2 + bx + c = 0,$$

where x is a variable and a, b, c are constants, is called the *general equation of the second degree in one variable,* or, a *quadratic equation in x.*

75. Methods for Solving the Equation $ax^2 + bx + c = 0$.

1. *Factoring by Inspection.* When the trinomial $ax^2 + bx + c$ can readily be factored, then each factor, equated to zero, gives a value of x.

Example. $x^2 - 5x + 6 = 0,$

or $(x - 2)(x - 3) = 0.$

$\therefore \quad x - 2 = 0 \quad \text{or} \quad x - 3 = 0.$

Hence $x = 2 \quad \text{or} \quad x = 3.$

2. *By Completing the Square.*

(a) Given $ax^2 + bx + c = 0.$

Transpose c: $ax^2 + bx = -c.$

Divide by a: $x^2 + \frac{b}{a}x = -\frac{c}{a}.$

Add $\left(\frac{b}{2a}\right)^2$ to both members:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2,$$

or

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

$$\therefore \quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{1}{2a}\sqrt{b^2 - 4ac}.$$

Hence $\qquad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

(b) Given $\qquad ax^2 + bx + c = 0.$

Transpose c: $\qquad ax^2 + bx = -c.$

Multiply by $4a$: $\qquad 4a^2x^2 + 4abx = -4ac.$

Add b^2: $\qquad 4a^2x^2 + 4abx + b^2 = b^2 - 4ac,$

or $\qquad (2ax + b)^2 = b^2 - 4ac.$

$\qquad \therefore \quad 2ax + b = \pm\sqrt{b^2 - 4ac}.$

Hence $\qquad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

Example. $\qquad 2x^2 + x - 6 = 0.$

(a) Transpose -6: $\qquad 2x^2 + x = 6.$

Divide by 2: $\qquad x^2 + \frac{1}{2}x = 3.$

Add $(\frac{1}{4})^2$: $\qquad x^2 + \frac{1}{2}x + (\frac{1}{4})^2 = 3 + (\frac{1}{4})^2,$

or $\qquad (x + \frac{1}{4})^2 = \frac{49}{16}.$

$\qquad \therefore \quad x + \frac{1}{4} = \pm\frac{7}{4},$

and $\qquad x = \pm\frac{7}{4} - \frac{1}{4} = \frac{3}{2}$ or $-2.$

(b) Transpose -6: $\qquad 2x^2 + x = 6.$

Multiply by 8: $\qquad 16x^2 + 8x = 48.$

Add 1^2 or 1: $\qquad 16x^2 + 8x + 1 = 49,$

or $\qquad (4x + 1)^2 = 49.$

$\qquad \therefore \quad 4x + 1 = \pm 7.$

Hence $\qquad x = \frac{3}{2}$ or -2, as before.

3. *By Formula.* By completing the square according to either method, we obtained

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Any quadratic equation in x may be solved directly by means of this formula, by merely inserting for a, b, and c their values from the given equation. *The formula should be carefully committed to memory.*

Definition. A *root* of an equation is a value of the variable which satisfies the equation.

By the formula, the two roots of any quadratic equation can be obtained.

Example. $2x^2 + x - 6 = 0.$

$$x = \frac{-1 \pm \sqrt{1 - 4 \times 2 \times (-6)}}{4} = \frac{-1 \pm 7}{4} = \frac{3}{2} \text{ or } -2.$$

76. Exercises. — Solve for x:

1. $x^2 + 4x - 21 = 0.$
2. $x^2 - 8x = -12.$
3. $x^2 + 6x = 7.$
4. $x^2 + 12 = 7x.$
5. $24 = x^2 - 5x.$
6. $5x^2 + 3x - 2 = 0.$
7. $3x^2 - 5x - 42 = 0.$
8. $3x^2 - 50 = 25x.$
9. $2x^2 + 13x = -15.$
10. $3x^2 - 7x - 6 = 0.$
11. $5x(x-2) + \frac{1}{4} = 1 - 3x.$
12. $(1-3x)(x-6) = 2(x+2).$
13. $(x+1)(2x+3) = 4x^2 - 22.$
14. $7x^2 - 48 = 2x(x-7).$
15. $13x^2 - 30 = 6(1+x)^2 + 63.$
16. $ax - b = x^2.$
17. $bx - 2b^2 - x^2 = 2bx.$
18. $x^2 + mn = (m+n)x.$
19. $cx + 2c - x^2 = -2x.$
20. $x^2 - \frac{ax}{2} = \frac{a^2}{2}.$

77. Nature of the roots of the equation $ax^2 + bx + c = 0.$ — The values of x obtained by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

will be

1. **real and unequal** if $b^2 - 4ac > 0;$
2. **real and equal** if $b^2 - 4ac = 0;$
3. **imaginary** if $b^2 - 4ac < 0.$

For, in the first case the radicand in the formula for x is positive, hence the square roots are real; in the second case the radicand vanishes, and the two values of x reduce to the common value $-b \div 2a$; in the third case the radicand is negative, hence both square roots are imaginary.

The expression $b^2 - 4ac$, on whose value depends the nature of the roots, is called the **discriminant** of the equation $ax^2+bx+c=0.$

When the discriminant vanishes, the roots are equal; $ax^2 + bx + c$ *is then a perfect square.*

Exercises. — Without solving the equations, determine the nature of the roots of:

1. Exercises 1–10 of (76).
2. $4x^2 + 4x + 1 = 0.$
3. $x^2 + x + 1 = 0.$
4. $6x^2 + 2x - 1 = 0.$
5. $9x^2 + 12x + 4 = 0.$
6. $\frac{1}{2}x^2 - \frac{1}{3}x - \frac{1}{4} = 0.$
7. $0.1x^2 + 0.5x + 0.8 = 0.$
8. $1\frac{1}{2}x^2 - 6\frac{1}{3}x + 8\frac{1}{4} = 0.$
9. $\frac{1}{4}x^2 - \frac{1}{3}x + \frac{1}{9} = 0.$
10. $0.06x^2 + 0.22x + 0.08 = 0.$

For what real values of the literal quantity involved in the following equations will the roots be real and unequal, equal, or imaginary respectively: indicate your answer on a diagram.

Example. $2x^2 - 3cx + 5c = 0.$

For what values of c will the roots of this equation be real and unequal? Equal? Imaginary?

Solving:

$$x = \frac{3c \pm \sqrt{9c^2 - 40c}}{4}.$$

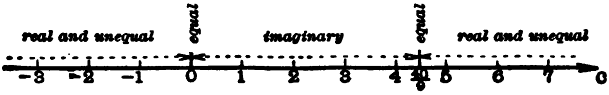
$$\therefore \quad \text{roots are} \begin{cases} \text{real and unequal} \\ \text{equal} \\ \text{imaginary} \end{cases} \quad \text{according as } 9c^2 - 40c \begin{matrix} > \\ = \\ < \end{matrix} 0.$$

But $9c^2 - 40c = 0$ when $c = 0$ or $c = \frac{40}{9}$. Hence for these values the roots are equal.

To make the roots real and unequal we must have $9c^2 - 40c > 0$ or $9c^2 > 40c$. This will be true if c is any negative number whatever, or if c is any positive number greater than $\frac{40}{9}$.

For imaginary roots we must have $9c^2 - 40c < 0$, which will be true for values of c between 0 and $\frac{40}{9}$.

All the above results are indicated in the following diagram:



11. $x^2 + 2x + c = 0.$
12. $4x^2 + 4x + h = 0.$
13. $3x^2 - 2x - k = 0.$
14. $\frac{1}{2}x^2 - \frac{1}{3}x + 4a = 0.$
15. $x^2 + 2bx + 4 = 0.$
16. $3x^2 - 4kx + 5 = 0.$
17. $6x^2 + \frac{4}{m}x - 3 = 0.$
18. $2x^2 + 4hx - h^2 = 0.$
19. $2x^2 + 4ax - a = 0.$
20. $ax^2 + 2x + 1 = a.$
21. $a^2x^2 + ax + 5 = 0.$
22. $2cx^2 + 3x - c^2 = 0.$
23. $(1 + k)x^2 + x - k = 0.$
24. $\frac{x^2}{n} + \frac{1}{2}x + \frac{1}{n+1} = 0.$

78. Relations between the Coefficients and Roots of a Quadratic Equation. — The roots of the equation

$$ax^2 + bx + c = 0$$

are:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a};$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Hence

$$x_1 + x_2 = -\frac{b}{a},$$

and

$$x_1 x_2 = \frac{c}{a}.$$

That is, *if the equation be divided by the coefficient of x^2, the new coefficient of x, with its sign changed, equals the sum of the roots; the new constant term equals the product of the roots.*

79. Factors of the Trinomial $ax^2 + bx + c$. — If x_1 and x_2 be the roots of the equation $ax^2 + bx + c = 0$, the trinomial is divisible by $x - x_1$ and $x - x_2$. See (*f*) of (**11**). But

$$(x - x_1)(x - x_2) = x^2 - (x_1 + x_2)x + x_1x_2$$
$$= x^2 + \frac{b}{a}x + \frac{c}{a}.$$

Multiplying through by a,

$$a(x - x_1)(x - x_2) = ax^2 + bx + c.$$

Hence *to factor the trinomial $ax^2 + bx + c$, place it equal to zero and solve for x; subtract each root from x, form the product of these differences and multiply it by a.*

80. Exercises.

1. Find the sum and the product of the roots of the equations in exercises 1–10 of (**76**). Write each equation in factored form.

Form equations whose roots are:

2. 2, 3; 4, −1; −2, −1.
3. $a, 2a$; p, q; $m + n, m - n$.
4. $\sqrt{-1}, -\sqrt{-1}$; $1 + \sqrt{-1}, 1 - \sqrt{-1}$; $a + b\sqrt{-1}, a - b\sqrt{-1}$.

81. Graphic Solution of Quadratic Equations. — In order to solve the equation

$$ax^2 + bx + c = 0,$$

we must find the values of x which reduce the trinomial $ax^2 + bx + c$ to zero. When a, b, c are given numerical values, the required values of x, when real, may be obtained, exactly or approximately, by trial.

Consider, for example, the equation

$$x^2 - 2x - 2 = 0.$$

Designate the trinomial on the left by y, so that

$$y = x^2 - 2x - 2.$$

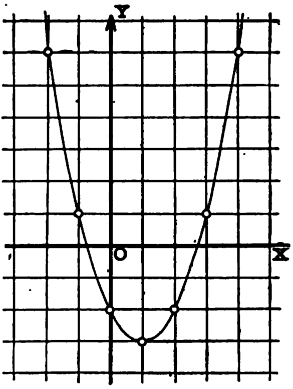
Now form a table showing the values of y corresponding to a series of assumed values of x:

$$x = \cdots -2, \quad -1, \quad 0, \quad +1, \quad +2, \quad +3, \quad +4, \cdots;$$
$$y = \cdots +6, \quad +1, \quad -2, \quad -3, \quad -2, \quad +1, \quad +6, \cdots.$$

We see that $y = 0$ for some value of x between -1 and 0. Also, y must be zero again for a value of x between $+2$ and $+3$, hence the other root lies between 2 and 3.

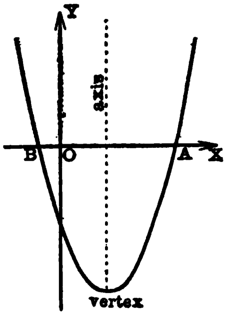
Now consider the pairs of corresponding values of x and y as the coördinates of a series of points and draw a smooth curve through them (figure). Scaling off the values of x for which $y = 0$, we have

$x = -0.7$ and $x = 2.7$ approximately.



$y = x^2 - 2x - 2.$

82. Parabola. — The curve in the figure is called a **parabola.** It is an example of a class of curves all of which have similar forms. The point where the curve bends most sharply is its **vertex,** and a line through the vertex and dividing the curve into two symmetrical portions is called the **axis** (figure). The segments OA and OB, measured from the origin to the points where the curve cuts the x-axis, are called the **x-intercepts.** The intercepts are positive when extending to the right from O, negative when extending to the left.



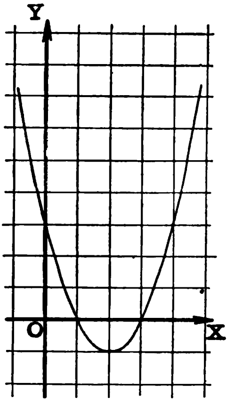
It will be found that **the graph of the equation**

$$y = ax^2 + bx + c$$

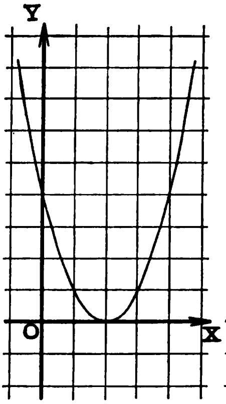
is always a parabola, with its axis parallel to the y-axis. (We assume $a \neq 0$.)

The parabola will cut the x-axis in two distinct points, or be tangent to the x-axis, or will not cut the x-axis at all according as the equation $ax^2 + bx + c = 0$ has real and unequal, or equal, or imaginary roots. For in the first case y is zero for two distinct values of x, in the second for two equal values of x, and in the third for no real value of x.

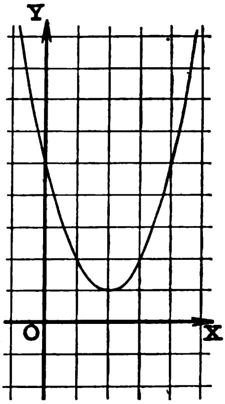
These three cases are illustrated in the figures below.



$y = x^2 - 4x + 3$
$b^2 - 4ac > 0$



$y = x^2 - 4x + 4$
$b^2 - 4ac = 0$



$y = x^2 - 4x + 5$
$b^2 - 4ac < 0$

83. Exercises.

1. Solve graphically several of the equations in exercises 1–10 of (76).

2. On the same diagram construct the graphs of $y = x^2 + 2x$, $y = x^2 + 2x + 1$, and $y = x^2 + 2x + 2$.

3. As in Ex. 2 for $y = -x^2 - 2x$, $y = -x^2 - 2x - 1$, and $y = -x^2 - 2x - 2$.

4. What is the effect on the graph of $y = ax^2 + bx + c$ when c is increased or diminished?

5. What is the effect on the graph of changing the signs of all terms of the trinomial?

84. Equations Reducible to Quadratics.

Example 1. $2x^4 - 7x^2 + 6 = 0.$

Solve for x^2 as the unknown quantity.

$$x^2 = \frac{7 \pm \sqrt{49 - 48}}{4} = 2 \text{ or } \frac{3}{2}.$$

$$x = \pm\sqrt{2} \text{ or } \pm\sqrt{\frac{3}{2}}.$$

Example 2. $x^{-\frac{2}{3}} - 8x^{-\frac{1}{3}} = 9.$

Solve for $x^{-\frac{1}{3}}$ as the unknown.

$$x^{-\frac{1}{3}} = \frac{8 \pm 10}{2} = 9 \text{ or } -1.$$

$$x = \frac{1}{729} \text{ or } -1.$$

Example 3. $(2x^2 + 5x)^2 - 6 = 2x^2 + 5x$.
Solve for $2x^2 + 5x$ as the unknown.

$$(2x^2 + 5x)^2 - (2x^2 + 5x) - 6 = 0.$$

$$2x^2 + 5x = \frac{1 \pm 5}{2}$$

$$= 3 \text{ or } -2.$$

$$2x^2 + 5x = 3$$

$$\text{or } 2x^2 + 5x = -2.$$

$$x = \tfrac{1}{2} \text{ or } -3;$$

$$\text{or } x = -\tfrac{1}{2} \text{ or } -2.$$

Exercise. Verify the answers in the above examples by substitution.

Example 4. $x + \sqrt{2x^2 + 1} = 1.$

Transpose: $\sqrt{2x^2 + 1} = 1 - x.$

Square and collect terms: $x^2 + 2x = 0.$

Therefore $x = 0$ or -2.

Example 5. $x - \sqrt{2x^2 + 1} = 1.$

Transpose: $-\sqrt{2x^2 + 1} = 1 - x.$

Square, etc.: $x^2 + 2x = 0.$

Therefore $x = 0$ or -2, as in example 4.

Exercise. Verify the answers in examples 4 and 5.

On substituting the values found in examples 4 and 5 in the given equations, we find that the first equation is satisfied by both values of x, but not the second, provided we assume, as usual, that $\sqrt{2x^2 + 1}$ stands for the positive square root.

The equation of example 5 may be put in the form

$$x - 1 = \sqrt{2x^2 + 1}.$$

Evidently these two expressions are not equal to each other for *any* real value of x. For, if x be less than 1, they are of unlike signs; if x be greater than 1, $\sqrt{2x^2}$ is certainly greater than x, and therefore $\sqrt{2x^2 + 1} > x - 1$. Hence the solution of example 5 as above has led to incorrect results.

The reason for this is that on squaring in the second step of the solution the sign of the radical disappears, and from that point on we are really solving example 4 also.

When an equation is squared to clear of radicals, the answers should be carefully verified and only those retained which satisfy the given condition.

85. Exercises and Problems.

1. $\sqrt{x^2+x-7}=\sqrt{8x-7}.$
2. $\sqrt{5x^2-10x+6}=\sqrt{3(5x+2)}.$
3. $\sqrt{2x^2-x-2}=\sqrt{x+2}.$
4. $4x+3=\sqrt{7x^2+12x+9}.$
5. $\sqrt{x+2}+\sqrt{x+7}=5\sqrt{x-1}.$
6. $\sqrt{2x+3}+\sqrt{7x-20}=\sqrt{3x+7}.$
7. $\sqrt{x+2}+\sqrt{3x-6}=10.$
8. $\sqrt{2x+17}+\sqrt{2x-4}=\frac{7}{2}\sqrt{x}.$
9. $\sqrt{2x+3}+\sqrt{x-2}=2\sqrt{x+1}.$
10. $\sqrt{10+x}=\sqrt{7x-6}-2.$
11. $\dfrac{\sqrt{x^2-5x+8}}{\sqrt{2x^2+x-20}}=\dfrac{1}{2}.$
12. $\dfrac{\sqrt{3x^2+9x+11}}{\sqrt{4x^2+7x+4}}=\dfrac{3}{2}.$
13. $\dfrac{\sqrt{9x^2-12x+4}}{\sqrt{18x^2-39x+19}}=\dfrac{3}{2}.$
14. $\dfrac{\sqrt{9x^2-12x+4}}{\sqrt{18x^2-39x+19}}=-\dfrac{3}{2}.$
15. $\dfrac{2}{y+\sqrt{1-y^2}}+\dfrac{2}{y-\sqrt{1-y^2}}=\dfrac{12}{7}.$
16. $\dfrac{5}{2x+\sqrt{4x^2+5}}-\dfrac{5}{2x-\sqrt{4x^2+5}}=6.$
17. $\dfrac{2z-1}{\sqrt{2z}+1}-1=\dfrac{\sqrt{2z}-1}{2}.$
18. $\dfrac{2v-1}{\sqrt{v+1}}=\dfrac{7\sqrt{2v}}{2}-3\sqrt{v+1}.$
19. $\dfrac{36}{\sqrt{2u+1}}=\sqrt{2u}+\sqrt{2u+1}.$
20. $\sqrt{7t-3}+\dfrac{11t+4}{\sqrt{4t-7}}=7\sqrt{4t-7}.$
21. $\dfrac{\sqrt{5x^2+4}-\sqrt{3x^2-2}}{\sqrt{5x^2+4}+\sqrt{3x^2-2}}=\dfrac{1}{6}.$
22. $\dfrac{\sqrt{3x^2+4}+\sqrt{x^2+5}}{\sqrt{3x^2+4}-\sqrt{x^2+5}}=7.$
23. $\dfrac{\sqrt{5x+1}+\sqrt{4-x}}{\sqrt{5x+1}-\sqrt{4-x}}=\dfrac{2\sqrt{x+1}+1}{2\sqrt{x+1}-1}.$

(Reduce by composition and division.)

24. $\dfrac{2}{x}=\dfrac{3}{\sqrt{1+2x^2}}-\dfrac{2}{1+x}.$
25. $2\left(1+\dfrac{3}{x}\right)+3\sqrt{\dfrac{x+3}{x}}=27.$
26. $\dfrac{\sqrt{x^2+9}}{\sqrt{x+7}}+\sqrt{x-7}=\dfrac{8}{\sqrt{x-7}}.$
27. $\dfrac{x-m^2}{x+m^2}=\dfrac{p^2+x}{p^2-x}.$
28. $\dfrac{n^2-x^2}{n^2+x^2}=\dfrac{x^2+p^2}{x^2-p^2}.$

29. $\frac{a^2}{x} - \frac{x}{b^2} = \frac{x}{a^2 - b^2}.$

30. $\frac{a - x}{1 - ax} = \frac{1 - cx}{c - x}.$

31. $\frac{x^5 + a^5}{x + a} - \frac{x^5 - a^5}{x - a} = 6\,a^3x.$

32. $\frac{\sqrt{a^2 - x^2}}{x} = \frac{x}{a} - \frac{a}{x}.$

33. $\frac{y^4 - m^4}{y - m} + \frac{y^4 + m^4}{y + m} = 2\,y^3.$

34. $\sqrt{a + 2\,x} - \sqrt{a - 2\,x} = \sqrt{a}.$

35. $\frac{m - \sqrt{2\,my - y^2}}{m + \sqrt{2\,my - y^2}} = a.$

36. $\sqrt{x} + \sqrt{a - x} = \frac{a}{2\sqrt{x}}.$

37. $\frac{\sqrt{x} + \sqrt{b}}{\sqrt{x} - \sqrt{b}} = \frac{\sqrt{a - x} + \sqrt{b - x}}{\sqrt{a - x} - \sqrt{b - x}}.$

38. $\sqrt{x} + \sqrt{a - \sqrt{ax + x^2}} = \sqrt{a}.$

39. $\sqrt{a - x} + \sqrt{-(a^2 + ax)} = \frac{a}{\sqrt{a - x}}.$

40. $\frac{\sqrt{a - x} + \sqrt{x - b}}{\sqrt{a - x} - \sqrt{x - b}} = \sqrt{\frac{a - x}{x - b}}.$

41. $x\sqrt[3]{x} + 5\,x\sqrt[3]{\frac{1}{x}} = 36.$

42. $\frac{a - x}{\sqrt{a - x}} + \frac{x - b}{\sqrt{x - b}} = \sqrt{a - b}.$

43. $\sqrt{\frac{a - x}{b + x}} + \sqrt{\frac{b + x}{a - x}} = c.$

44. $\sqrt{\frac{a - x}{b - x}} - \sqrt{\frac{b - x}{a - x}} = c.$

45. $x^{\frac{10}{3}} - 16x^{\frac{5}{3}} = 512.$

46. $x^{2m} + 3\,ax^m = 4\,a^2.$

47. $x^2 + \sqrt{5\,x + x^2} = 42 - 5\,x.$

48. $\sqrt[3]{x} - 5\sqrt[3]{x^2} = -18.$

49. $7\sqrt[3]{-x} + \sqrt[3]{x^2} = -12.$

50. $x^2 + 12 = 7\,x - \sqrt{x^2 - 7\,x + 18}.$

51. $\sqrt[m]{(1 + x)^2} - \sqrt[m]{(1 - x)^2} = \sqrt[m]{1 - x^2}.$

52. Find three consecutive integers, the sum of whose squares is 1085.

53. Find three consecutive odd integers, the sum of whose squares is 1331.

54. The sum of the squares of three consecutive integral multiples of 4 is 3920. Find the numbers.

55. A rectangle, twice as long as it is wide, has an area of 2450 square feet. Find its dimensions.

56. How large a square must be cut from each corner of a rectangular card 6 × 12 inches so that the remaining piece shall contain 47 square inches?

57. As in 56, except that the original dimensions are $a \times b$ inches and the remaining area A square inches.

58. What changes must be made in the dimensions of a rectangle 2 × 12 inches to double the area without changing the perimeter?

59. As in 58, when the original dimensions are $a \times b$ inches.

60. State some values of a and b for which exercise 59 is impossible.

61. Find the radius of a cylinder whose height is 10 feet, if the total surface in square feet must equal the volume in cubic feet.

62. As in 61, except that total surface equals twice the volume.

63. As in 61, except that total surface equals n times the volume. For what values of n is the problem impossible?

64. A cylindrical can 10 inches high has a total surface, S, of $150\frac{6}{7}$ square inches, including top and bottom. Find its radius, r. Find r when $S = 100$ square inches. (Take $\pi = \frac{22}{7}$.)

65. If a quantity of water whose volume is 1000 c.c. at temperature 0° C. is heated to $x°$, its volume will be

$$v = 1000 - .058\,x + .0076\,x^2.$$

At what temperature will the volume be 1001 c.c.? 1002 c.c.?

Ans. $x = 16° \pm;\ 21° \pm$.

66. If a projectile is thrown vertically downward with a speed of v feet per second it will fall, in t seconds, through a distance (in feet) approximately equal to

$$d = vt + 16\,t^2.$$

If the projectile is thrown from an aeroplane 3000 feet above ground with a speed of 40 feet per second, in how many seconds will it reach the ground?

67. If a projectile is thrown upward with a speed of v feet per second its height (in feet) in t seconds will be

$$h = vt - 16\,t^2.$$

If $v = 100$, when will the height be 100 feet? Explain the two answers. Will the height reach 200 feet? What will be the greatest height reached and in how many seconds? (Find what value of h gives equal values for t.)

68. If a gun is pointed at an angle of 45° with the horizontal and a projectile leaves the muzzle with a speed of v feet per second, the height, y, of the projectile above the ground is connected with its horizontal distance, x, from the gun, by the equation

$$y = x - \frac{16\,x^2}{v^2}.$$

(Here x and y are in feet.) Find x when $v = 1000$, and $y = 2000$. Explain the two answers. Find the maximum value of y when $v = 1000$. (Find the value of y for which the values of x will be equal.)

69. For what real values of x is $x^2 + 10\,x + 9$ positive? zero? negative? (Graph.)

70. Show that $6 + 2\,a + a^2$ cannot be negative if a is real. (Graph.)

71. Show that $3\,a - a^2 - 5$ cannot be positive if a is real. (Graph.)

SIMULTANEOUS QUADRATICS

86. Definition. — The *degree of a monomial* involving one or more literal quantities is the sum of the exponents of such literal quantities as may be specified.

For example $a^p x^m y^n$ is of degree m in x, n in y, $m + n$ in x and y, $m + n + p$ in a, x, and y.

The *degree of a polynomial* is that of its term of highest degree.

A *quadratic equation in several variables* is one in which all the variable terms are of the first or second degree, at least one term of the second degree being actually present.

87. Solution of Two Simultaneous Equations in Two Variables, one being Linear, the other Quadratic. — The most general forms of such equations are:

$$(1) \qquad px + qy + r = 0,$$
$$(2) \qquad ax^2 + by^2 + cxy + dx + ey + f = 0.$$

Solution.

1. Solve (1) for one of the variables in terms of the other. Thus:

$$y = -\frac{px + r}{q}.$$

2. Substitute this value in (2), obtaining a quadratic equation in x.

3. Solve this quadratic for x, and let its roots be x_1 and x_2.

4. The corresponding values of y are now found by substituting these values for x in the first step. Thus:

$$y_1 = -\frac{px_1 + r}{q}, \quad \text{and} \quad y_2 = -\frac{px_2 + r}{q}.$$

Example. $(1) \quad x + y = 1,$

$(2) \quad x^2 + y^2 = 4.$

From (1), $y = 1 - x.$

Substituting in (2): $x^2 + (1 - x)^2 = 4$

or $2x^2 - 2x - 3 = 0.$

Hence $x_1 = \frac{1}{2} + \frac{1}{2}\sqrt{7}$; $x_2 = \frac{1}{2} - \frac{1}{2}\sqrt{7}.$

Then $y_1 = \frac{1}{2} - \frac{1}{2}\sqrt{7}$; $y_2 = \frac{1}{2} + \frac{1}{2}\sqrt{7}.$

Reducing to decimals, we have approximately

$$(x_1, y_1) = (+1.8, -0.8) \quad \text{and} \quad (x_2, y_2) = (-0.8, +1.8).$$

In this case there are two distinct real solutions.

88. Nature of the Solutions of Equations (1) and (2) of (87). — The values x_1 and x_2 obtained in the third step of the solution in (87) are either real and unequal, real and equal, or both imaginary. Then the values of y obtained in the fourth step will be of the same nature as the values of x.

Hence there are always two solutions, which may be real and unequal, real and equal, or imaginary.

These three cases may be illustrated by means of the equations,

$$(1) \qquad x + y = k,$$
$$(2) \qquad x^2 + y^2 = 4.$$

Then $\quad x^2 + (k - x)^2 = 4,$

or $\quad 2x^2 - 2kx + (k^2 - 4) = 0.$

Hence $\quad x_1 = \frac{1}{2}\left(k + \sqrt{8 - k^2}\right)$

and $\quad x_2 = \frac{1}{2}\left(k - \sqrt{8 - k^2}\right).$

$\quad y_1 = \frac{1}{2}\left(k - \sqrt{8 - k^2}\right)$

and $\quad y_2 = \frac{1}{2}\left(k + \sqrt{8 - k^2}\right).$

These solutions will be

real and unequal if $k^2 < 8$;
real and equal if $k^2 = 8$;
imaginary if $k^2 > 8$.

89. Graphic Solution. — Consider the equations

$$(1) \qquad x + y = 1,$$
$$(2) \qquad x^2 + y^2 = 4.$$

Considering x and y as the coördinates of a variable point, all values of x and y which satisfy equation (1) give rise to a series of points lying on a straight line (figure on next page).

Let us now mark some points whose coördinates satisfy equation (2), which we put into the form

$$y = \pm\sqrt{4 - x^2}.$$

Assuming a set of values for x, and calculating the corresponding values of y, we have

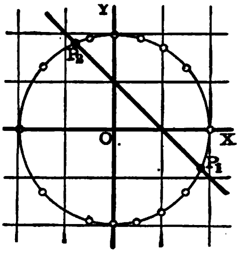
$$x = 0, \quad \tfrac{1}{2}, \quad 1, \quad 1\tfrac{1}{2}, \ 2, \ 2\tfrac{1}{2}, \ \cdots;$$
$$y = \pm 2, \ \pm\tfrac{1}{2}\sqrt{15}, \ \pm\sqrt{3}, \ \pm\tfrac{1}{2}\sqrt{7}, \ 0, \ \text{imaginary}.$$

For negative values of x we obtain the same values of y over again.

On plotting these values we obtain a series of points all of which lie on a *circle* of radius 2, center at the origin.

The points of intersection of the line and the circle have coördinates which satisfy both equations at once, and are therefore the required solutions. Scaling them off from the figure we have

$$(x_1, y_1) = (1.8, -0.8) \quad \text{and}$$
$$(x_2, y_2) = (-0.8, +1.8), \text{ as in } (\mathbf{87}).$$

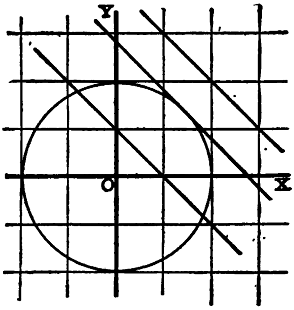


Straight line $x + y = 1$
Circle $x^2 + y^2 = 4$

90. Graphic Illustration of the Three Cases of (88). — In (1) of (**88**), let us put successively $k = 1, 2\sqrt{2}$, and 4, so that $k^2 < 8$, $= 8$, and > 8 respectively. We have then the equations,

$$(1) \qquad x + y = 1; \qquad x + y = 2\sqrt{2}; \qquad x + y = 4,$$
$$(2) \qquad x^2 + y^2 = 4; \qquad x^2 + y^2 = 4; \qquad x^2 + y^2 = 4.$$

The three straight lines and the circle are shown in the adjacent figure. When $k = 1$, the line cuts the circle in two distinct points; when $k = 2\sqrt{2}$, the line is tangent to the circle; when $k = 4$, the line fails to meet the circle. We may consider these three cases as arising from special positions of a variable line which moves parallel to itself and occupies in turn the positions of the three lines in the figure.

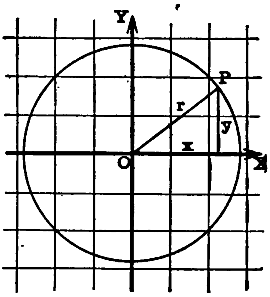


Circle $x^2 + y^2 = 4$

91. Standard Equation of the Circle.— The equation

$$x^2 + y^2 = r^2$$

is satisfied by the coördinates of every point on a circle of radius r, center at the origin, and by no other point. It is called *the standard equation of the circle.*



$x^2 + y^2 = r^2$
Circle, radius r, center at origin

Exercises. Solve for x and y, and check carefully by graphs.

1. $\begin{cases} x^2 + y^2 = 1, \\ x - y = 0. \end{cases}$
2. $\begin{cases} x^2 + y^2 = 1, \\ x - y = 2. \end{cases}$
3. $\begin{cases} x^2 + y^2 = 1, \\ x - y = \sqrt{2}. \end{cases}$
4. $\begin{cases} x^2 + y^2 = 4, \\ 2x + y = 2. \end{cases}$
5. $\begin{cases} x^2 + y^2 = 4, \\ x - 2y = 3. \end{cases}$
6. $\begin{cases} x^2 + y^2 = 16, \\ 2x - 3y = 4. \end{cases}$
7. $\begin{cases} x^2 + y^2 = 9, \\ 3x + 4y = 12. \end{cases}$
8. $\begin{cases} x^2 + y^2 = 9, \\ 4x - 5y = 20. \end{cases}$
9. $\begin{cases} 4x^2 + 4y^2 = 1, \\ 3x - y = 1. \end{cases}$

10. Determine k so that the line $x + y = k$ shall be tangent to the circle $x^2 + y^2 = 4$. (*Suggestion.* Substitute the value of y from the first equation in the second, and arrange the result as a quadratic in x. Find what values of k will make the roots of this quadratic equal.)

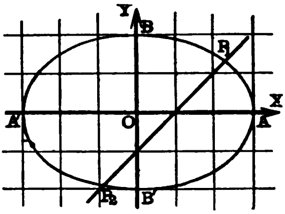
11. Determine m so that the line $y = mx + 5$ shall touch the circle $x^2 + y^2 = 5$.

12. As in 11, for the line $y = mx + 2$ and the circle $5x^2 + 5y^2 = 2$.

92. Ellipse. — Consider the equations

$$x - y = 1,$$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$



Ellipse $\quad \frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$

Straight line $\quad x - y = 1$

Proceeding as in (**87**), we obtain

$$x_1 = \frac{9 + 12\sqrt{3}}{13} = 2.3 -;$$

$$x_2 = \frac{9 - 12\sqrt{3}}{13} = -0.9 -;$$

$$y_1 = \frac{-4 + 12\sqrt{3}}{13} = 1.3 -;$$

$$y_2 = \frac{-4 - 12\sqrt{3}}{13} = -1.9 -.$$

Graphic Solution. — All values of x and y which satisfy the first equation are the coördinates of points on a straight line. We now plot a series of points whose coördinates satisfy the second equation, which we solve for y in terms of x and write in the form

$$y = \pm \tfrac{2}{3}\sqrt{9 - x^2}.$$

When $x = -3, \quad -2, \quad -1, \quad 0, \quad +1, \quad +2, +3,$

$y = \quad 0, \pm \tfrac{2}{3}\sqrt{5}, \ \pm \tfrac{4}{3}\sqrt{2}, \pm 2, \ \pm \tfrac{4}{3}\sqrt{2}, \pm \tfrac{2}{3}\sqrt{5}, \quad 0.$

On plotting these points and drawing a smooth curve through them we obtain the curve in the above figure, called an **ellipse.** The line $A'A$ is called the **major axis** of the ellipse, $B'B$ the **minor axis,** and O is the **center.** In this case, $A'A = 6$ and $B'B = 4$; $OA = 3$ and $OB = 2$.

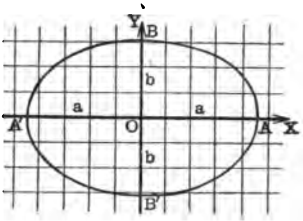
Scaling off the coördinates of the points of intersection of the two graphs, we have as our graphic solution

$$(x_1, y_1) = (2.3, 1.3); \quad (x_2, y_2) = (-0.9, -1.9).$$

93. Standard Equation of the Ellipse. — Every equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

represents an **ellipse,** whose major axis is 2 a, minor axis 2 b, center at the origin. It is called the *standard equation of the ellipse.*



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Ellipse, semi-axes a and b respectively

Exercises. Solve and check by graphs:

1. $\begin{cases} \dfrac{x^2}{9} + \dfrac{y^2}{4} = 1, \\ x + y = 0. \end{cases}$

2. $\begin{cases} \dfrac{x^2}{9} + \dfrac{y^2}{4} = 1, \\ x + y = 5. \end{cases}$

3. $\begin{cases} \dfrac{x^2}{9} + \dfrac{y^2}{4} = 1, \\ x + y = \sqrt{13}. \end{cases}$

4. $\begin{cases} 4x^2 + 9y^2 = 36, \\ x + y = -\sqrt{13}. \end{cases}$

5. $\begin{cases} \dfrac{x^2}{4} + y^2 = 1, \\ x + 2y = 2. \end{cases}$

6. $\begin{cases} x^2 + 4y^2 = 4, \\ x - y = 3. \end{cases}$

7. $\begin{cases} 9x^2 + 16y^2 = 25, \\ 2x - 3y = 6, \end{cases}$

8. $\begin{cases} 2x^2 + 3y^2 = 12, \\ 3x + y = 2. \end{cases}$

9. $\begin{cases} 9x^2 + 4y^2 = 1, \\ x - y = 1. \end{cases}$

10. Determine k so that the line $x - y = k$ shall be tangent to the ellipse $x^2 + 4y^2 = 4$.

11. Determine m so that the line $y = mx + 3$ shall touch the ellipse $4x^2 + 9y^2 = 36$.

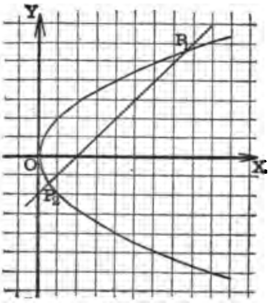
94. Parabola. — Consider the equations

$$x - y = 2,$$
$$y^2 = 4x.$$

Solving as in (**88**), we find

$$x_1 = 4 + 2\sqrt{3}, \quad x_2 = 4 - 2\sqrt{3},$$
$$y_1 = 2 + 2\sqrt{3}, \quad y_2 = 2 - 2\sqrt{3}.$$

The graphs are shown in the figure, that of the equation $y^2 = 4x$ being a **parabola,** whose vertex is at the origin and whose axis is the x-axis.



Parabola, $y^2 = 4x$

Exercise 1. Compare the graphic solution with that obtained by formula.

Exercise 2. For what value of k will the line $x - y = k$ be tangent to the parabola $y^2 = 4x$? Why are there not two values of k as in the exercise of (93)?

95. Standard Equations of the Parabola. — The equation

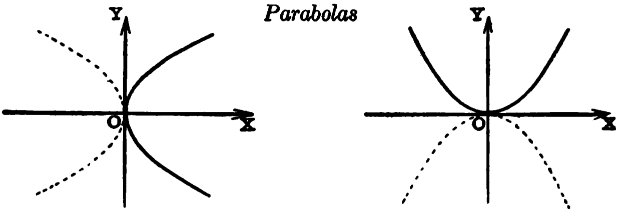
$$y^2 = 4ax$$

always represents a **parabola,** whose vertex is at the origin and whose axis is the x-axis. The curve extends to the right from O when a is positive, to the left when a is negative.

The equation

$$x^2 = 4ay$$

always represents a **parabola,** whose vertex is at the origin and whose axis is the y-axis. The curve extends upward when a is positive, downward when a is negative.



Parabolas

$y^2 = 4ax$ $x^2 = 4ay$

Exercises. Solve and check by graphs:

1. $\begin{cases} y^2 = x, \\ y = x. \end{cases}$

2. $\begin{cases} y^2 = 4x, \\ x + y = 1. \end{cases}$

3. $\begin{cases} 4y^2 = x, \\ 2x - y = 4. \end{cases}$

4. $\begin{cases} y^2 = -x, \\ y - x = 2. \end{cases}$

5. $\begin{cases} y^2 = -4x, \\ 3x + y = 3. \end{cases}$

6. $\begin{cases} x^2 = y, \\ y = 2x. \end{cases}$

7. $\begin{cases} x^2 = 4y, \\ x + 2y = 2. \end{cases}$

8. $\begin{cases} x^2 = -y, \\ 2x + 5y = 10. \end{cases}$

9. $\begin{cases} x^2 = -4y, \\ y - 2x = 1. \end{cases}$

10. Determine k so that the line $3x + y = k$ shall touch the parabola $y^2 + 4x = 0$.

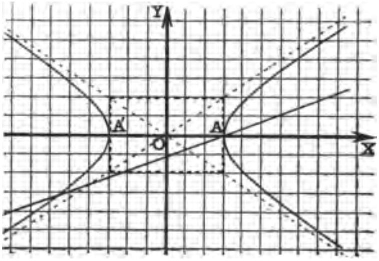
11. Determine m so that the line $y = mx + 2$ shall touch the parabola $y^2 = 8x$.

96. Hyperbola. — Consider the equations

$$3x - 8y = 9,$$

$$\frac{x^2}{9} - \frac{y^2}{4} = 1.$$

The graphs are shown in the figure.



Hyperbola, $\frac{x^2}{3^2} - \frac{y^2}{2^2} = 1$

Straight line, $3x - 8y = 9$

The graph of the second equation is a **hyperbola,** a curve consisting of two open branches which continually approach the diagonals, produced, of the dotted rectangle, but never cross them. These lines are called the **asymptotes** of the hyperbola. O is the center and $A'A$ the axis of the curve.

Exercise. Compare the solution of the given equations as obtained by formula with that from graph.

97. Standard Equation of the Hyperbola. — The equation

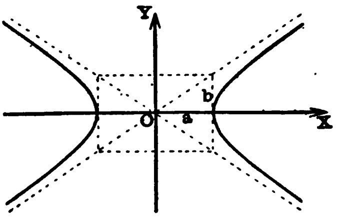
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

always represents a hyperbola whose axis coincides with the x-axis, and whose center is at the origin. The curve lies between its asymptotes, which are the diagonals, produced, of a rectangle whose sides are $2a$ and $2b$, parallel to the coördinate axes, with its center at the origin.

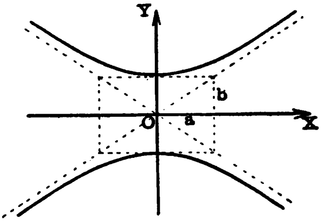
The equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$$

represents a hyperbola whose axis coincides with the y-axis.



Hyperbola, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$



Hyperbola, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$

Exercises. Solve and check by graphs:

1. $\begin{cases} x^2-y^2=1, \\ x-3y=1. \end{cases}$

2. $\begin{cases} x^2-y^2=1, \\ 2x-y=1. \end{cases}$

3. $\begin{cases} x^2-y^2=1, \\ \sqrt{5}x-y=2. \end{cases}$

4. $\begin{cases} \frac{x^2}{9}-\frac{y^2}{4}=4, \\ 5x+y=5. \end{cases}$

5. $\begin{cases} 4x^2-9y^2=36, \\ 4x+y=2. \end{cases}$

6. $\begin{cases} x^2-4y^2=4, \\ y=2x-6. \end{cases}$

7. $\begin{cases} 2x^2-3y^2=6, \\ 3x+y=6. \end{cases}$

8. $\begin{cases} x^2-y^2=-1, \\ y-3x=1. \end{cases}$

9. $\begin{cases} 4x^2-9y^2=-36, \\ 2y-x=0. \end{cases}$

10. Determine k so that the line $x-2y=k$ shall be tangent to the hyperbola $4x^2-9y^2=36$.

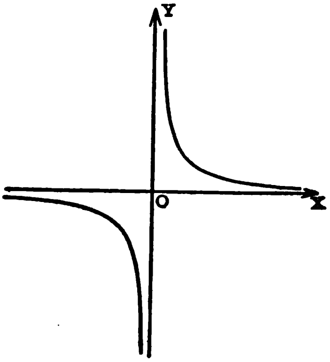
11. Determine m so that the line $y = mx - 2$ shall touch the hyperbola $x^2 - y^2 = 1$.

98. Rectangular Hyperbola.

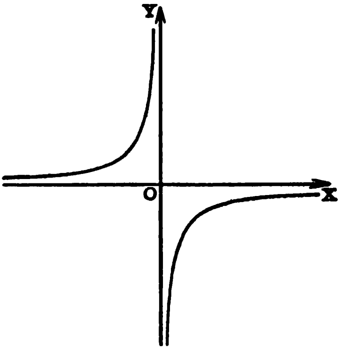
— The equation

$$xy = \pm k^2$$

always represents a hyperbola whose asymptotes are the coördinate axes; for the upper sign, its branches lie in the first and third quadrants, and for the lower sign in the second and fourth quadrants.



$xy = k^2$



$xy = -k^2$

99. The General Equation of the Second Degree,

$$ax^2 + by^2 + cxy + dx + ey + f = 0.$$

This includes all types of equation considered in the preceding sections and always represents one of the curves there shown, except in isolated cases when it can be factored into linear factors, in which case it represents a pair of straight lines, or when it is satisfied by the coördinates of single point only, as $x^2 + y^2 = 0$. The graph may also be imaginary, that is, the equation cannot be satisfied by any real values of x and y, as $x^2 + y^2 = -1$.

The curves represented by the general equation of the second degree are not restricted in position with respect to the coördinate axes as are those shown in the preceding figures. The center, vertices, axes and asymptotes may have any position whatever, depending on the numerical values of the coefficients a, b, c, d, e, f.

All curves represented by equations of the second degree in x and y may be obtained as plane sections of a circular cone. They are therefore called **conic sections.**

100. Exercises. — Give what facts you can about the curves represented by the following equations, without drawing the graphs: then draw the graphs.

1. $x^2 + y^2 = 9.$
2. $4x^2 + 4y^2 = 16.$
3. $3x^2 + 3y^2 = 15.$
4. $4x^2 + y^2 = 4.$
5. $x^2 + 4y^2 = 4.$
6. $16x^2 + 25y^2 = 400.$
7. $25x^2 + 16y^2 = 400.$
8. $2x^2 + 4y^2 = 9.$
9. $y^2 = 4x.$
10. $4y^2 = x.$
11. $x^2 = 4y.$
12. $4x^2 = y.$
13. $y^2 = -4x.$
14. $-4y^2 = x.$
15. $x^2 = -4y.$
16. $4x^2 = -y.$
17. $16x^2 - 25y^2 = 400.$
18. $16x^2 - 25y^2 = -400.$
19. $25x^2 - 16y^2 = 400.$
20. $25x^2 - 16y^2 = -400.$

Construct the graphs of the preceding equations on cross-section paper.
Construct the graphs of the equations:

21. $x^2 + y^2 - 6x - 8y = 0.$
22. $(x - y)^2 = 1.$
23. $3x^2+2xy+3y^2-16y+23=0.$
24. $x^2 - 5xy + 6y^2 = 0.$
25. $3x^2 + 2y^2 - 2x + y - 1 = 0.$
26. $x^2 + 2xy + y^2 = 0.$
27. $5x^2 + 2xy + 5y^2 = 0.$
28. $4xy + 6x - 8y + 1 = 0.$
29. $y^2 - xy - 5x + 5y = 0.$
30. $xy - y^2 = 1.$

Solve graphically and by formula several of the preceding equations with each of the equations

(a) $x - y = 1.$
(b) $2x + 3y = 6.$
(c) $x + y = 0.$
(d) $2x - y = 2.$

101. Solution of Two Simultaneous Quadratics. — When both quadratics are of the general form, as

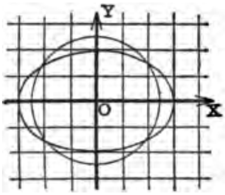
$$ax^2 + by^2 + cxy + dx + ey + f = 0,$$
$$a'x^2 + b'y^2 + c'xy + d'x + e'y + f' = 0,$$

they cannot usually be solved by elementary methods. For, if we solve one equation for y in terms of x say, and substitute in the other, we obtain, after rationalizing, an equation of the fourth degree in x. Such an equation requires rather complicated processes for its solution. We shall therefore leave aside the general case and discuss some special cases, such as usually arise in the practical application of algebra. We begin with some graphic illustrations.

102. Graphic Solution. — Since each of the above equations represents graphically a conic section, two such curves intersect in general in four points. All **real** solutions are shown by the intersections of the graphs, and may be read off, approximately at least, from the diagram.

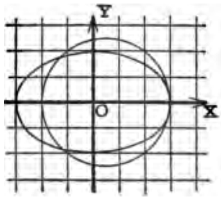
When the graphs intersect in less than four points (tangency is counted as two coincident points of intersection), some solutions are **imaginary** *or* **infinite.**

The various cases which may arise are illustrated in the figures below.



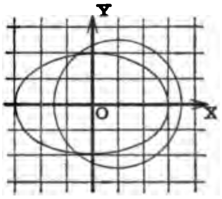
$$\frac{x^2}{9}+\frac{y^2}{4}=1$$
$$x^2+y^2=\tfrac{25}{4}$$

Four real solutions, all distinct.



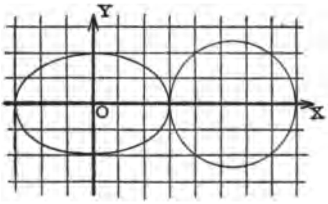
$$\frac{x^2}{9}+\frac{y^2}{4}=1$$
$$(x-\tfrac{1}{2})^2+y^2=\tfrac{25}{4}$$

Four real solutions, two being equal.



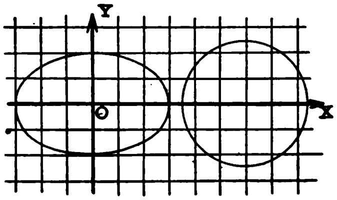
$$\frac{x^2}{9}+\frac{y^2}{4}=1$$
$$(x-1)^2+y^2=\tfrac{25}{4}$$

Two real distinct solutions, two imaginary.



$$\frac{x^2}{9}+\frac{y^2}{4}=1;\ \left(x-\frac{11}{2}\right)^2+y^2=\frac{25}{4}$$

Two real and equal solutions, two imaginary.

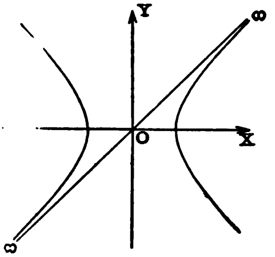


$$\frac{x^2}{9}+\frac{y^2}{4}=1;\ (x-6)^2+y^2=\frac{25}{4}$$

All four solutions imaginary.

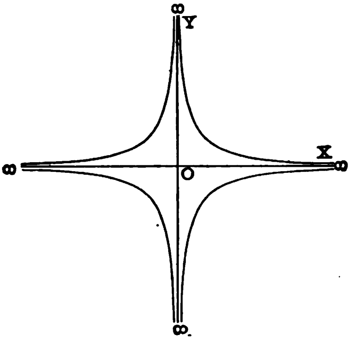
The student is urged to draw, or to picture to himself mentally as far as possible, graphs corresponding to all equations considered. He should be able to recognize at a glance the standard forms of equation of the conic sections.

Infinite solutions are illustrated in the two figures below:



$x^2 - y^2 = 1;\ x - y = 0$

Two solutions, both infinite.



$xy = 1;\ xy = -1$

Four solutions, all infinite.

We proceed to consider some special cases of simultaneous quadratic equations.

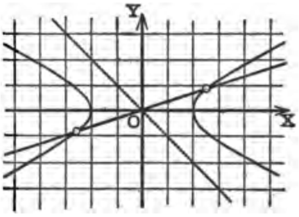
103. Case 1. Two quadratics, one of which is factorable.

Solution: *Factor the equation, put each factor equal to zero, and solve each of the resulting linear equations with the other quadratic.*

Rule for factoring a quadratic. Solve for y in terms of x (or x in terms of y); if the quantity under the radical is a perfect square the two values of y are of the form $y = ax + b$ and $y = a'x + b'$. The required factors are then

$$(y - ax - b)\,(y - a'x - b').$$

Graphically, the factorable quadratic represents a pair of straight lines, the other quadratic some conic. Each straight line may cut this conic in two real distinct points, in two real coincident points, or in two imaginary points (i.e. does not cut at all). Hence the four solutions may be all real and distinct, or equal in pairs, or imaginary in pairs.



Hyperbola, $x^2 - 4y^2 - 4 = 0$

Straight lines, $x^2 - 2xy - 3y^2 = 0$

or $x + y = 0$ *and* $x - 3y = 0$

Example 1. $\begin{cases} x^2 - 2xy - 3y^2 = 0, \\ x^2 - 4y^2 - 4 = 0. \end{cases}$

The factors of the first equation are, by inspection,

$$(x + y)\,(x - 3y) = 0.$$

$$\therefore \quad x + y = 0 \text{ or } x - 3y = 0.$$

Hence we have to solve

$$\begin{cases} x + y = 0, \\ x^2 - 4y^2 - 4 = 0, \end{cases}$$

and

$$\begin{cases} x - 3y = 0, \\ x^2 - 4y^2 - 4 = 0. \end{cases}$$

Solving the first pair, we have

$$(x_1, y_1) = \left(\frac{2}{\sqrt{-3}}, -\frac{2}{\sqrt{-3}}\right);$$

$$(x_2, y_2) = \left(-\frac{2}{\sqrt{-3}}, \frac{2}{\sqrt{-3}}\right).$$

These are imaginary. The line $x + y = 0$ does not cut the hyperbola (figure). Solving the second pair,

$$(x_3, y_3) = \left(\frac{6}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right); \quad (x_4, y_4) = \left(-\frac{6}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right).$$

These solutions are real, and the approximate values may be scaled off from the figure.

Note. An equation of the form $Ax^2 + Bxy + Cy^2 = 0$ *can always be factored.* Divide by the square of one of the variables, and solve for the ratio $\frac{y}{x}$ or $\frac{x}{y}$. Or, solve directly for y or x, regarding the equation as a quadratic in y or x.

The factors will be imaginary if $B^2 - 4AC < 0$, and in this case the graph of the equation is imaginary. In all other cases the graph is a pair of real straight lines, distinct if $B^2 - 4AC > 0$, and coincident if $B^2 - 4AC = 0$.

Example 2. Factor $2x^2 - 2xy + y^2 = 0$.

Divide by x^2:

$$\left(\frac{y}{x}\right)^2 - 2\left(\frac{y}{x}\right) + 2 = 0.$$

$$\therefore \quad \frac{y}{x} = 1 + \sqrt{-1} \text{ or } 1 - \sqrt{-1}.$$

Hence the factors are

$$[y - (1 + \sqrt{-1})x][y - (1 - \sqrt{-1})x] = 0.$$

If we solve the given equation for y, we get

$$y = \frac{2x \pm \sqrt{4x^2 - 8x^2}}{2} = x(1 \pm \sqrt{-1}).$$

The graph would be imaginary, because of the presence of the imaginary unit $\sqrt{-1}$. In this example we have $B^2 - 4AC = 4 - 8 = -4$.

Example 3.
$$\begin{cases} 2x^2 - y^2 - xy + 3y - 2 = 0. \\ x^2 - 4y = 0. \end{cases}$$

Solving the first equation for x in terms of y, we have

$$x = \frac{y \pm \sqrt{9y^2 - 24y + 16}}{4} = \frac{y \pm (3y - 4)}{4}.$$

Hence

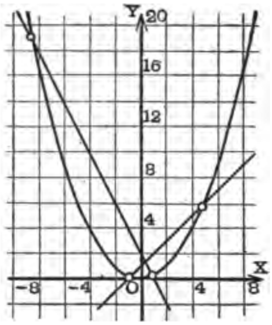
$$x - y + 1 = 0 \text{ or } 2x + y - 2 = 0.$$

Solving the first of these with the second equation above, we have

$$(x_1, y_1) = (2 + 2\sqrt{2}, 3 + 2\sqrt{2});$$
$$(x_2, y_2) = (2 - 2\sqrt{2}, 3 - 2\sqrt{2}).$$

From the second equation we obtain

$$(x_3, y_3) = (-4 + 2\sqrt{6}, 10 - 4\sqrt{6});$$
$$(x_4, y_4) = (-4 - 2\sqrt{6}, 10 + 4\sqrt{6}).$$



Parabola, $x^2 - 4y = 0$
Straight lines,
$2x^2 - y^2 - xy + 3y - 2$
or $x - y + 1 = 0$
and $2x + y - 2 = 0$

Exercises. Solve for x and y, and check graphically:

1. $\begin{cases} x^2 + y^2 = 1, \\ x^2 + yx - 2y^2 = 0. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 4, \\ x^2 - y^2 = 0. \end{cases}$

3. $\begin{cases} 4x^2 + 9y^2 = 36, \\ 2x^2 + 5xy + 3y^2 = 6x + 6y. \end{cases}$

4. $\begin{cases} x^2 - y^2 = 1, \\ xy - 2y + x = 2. \end{cases}$

5. $\begin{cases} y^2 - 4x = 0, \\ 6x^2 + xy - 12y^2 = 0. \end{cases}$

6. $\begin{cases} x^2 - 4y^2 = 4, \\ xy - 2y = 0. \end{cases}$

104. Case 2. Homogeneous equations.

Definition. An equation is called *homogeneous* when all of its variable terms are of the same degree. A constant term may be present. (In the further developments of mathematics, the last sentence is omitted from the definition.)

Two homogeneous quadratics have the forms

$$Ax^2 + Bxy + Cy^2 = D, \tag{1}$$
$$A'x^2 + B'xy + C'y^2 = D'. \tag{2}$$

Solution. Multiply the first equation by D', the second by D and subtract. The result is a new equation of the form

$$A''x^2 + B''xy + C''y^2 = 0, \tag{3}$$

which may be solved with either of the given equations by factoring, as in *Case* 1.

Graphically, equations (1) and (2) represent two conics, and equation (3) a third conic which consists of a pair of straight lines in case the factors are real. Conic (3) goes through the intersections of (1) and (2), since the coördinates of any point which satisfy (1) and (2) will also satisfy (3). Hence, when the factors of (3) are real, we obtain the intersections of (1) and (2) by finding the intersections of either of them with a pair of real straight lines. When these factors are distinct, there are two distinct lines, either of which may cut the conic in two real and distinct points, two coincident points, or two imaginary points. When the factors are imaginary the lines are imaginary, and all four solutions are imaginary.

Another method of solving two homogeneous equations in the forms (1) and (2) is to put in both of them $y = vx$. Then divide one equation by the other, and clear of fractions, after removing the common factor x^2. The result is a quadratic in v, whose roots we may represent by v_1 and v_2. Then

$$y = v_1x \text{ and } y = v_2x.$$

Substituting these values in turn in either of the given equations, we have two quadratic equations in x alone.

Example 1. $\begin{cases} 2x^2 - 3xy + 4 = 0, \\ 4xy - 5y^2 - 3 = 0. \end{cases}$

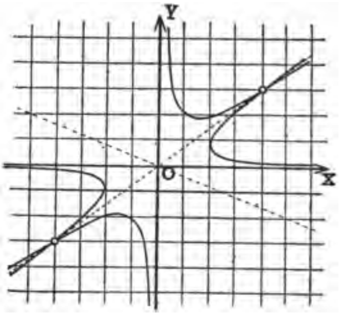
Transposing the constant terms we have

$$2x^2 - 3xy = -4.$$
$$4xy - 5y^2 = 3.$$

Multiplying the first equation by 3, the second by 4, and adding,

$$6x^2 + 7xy - 20y^2 = 0$$

or $\quad (3x - 4y)(2x + 5y) = 0.$



Hyperbolas, $\begin{cases} 2x^2 - 3xy + 4 = 0 \\ 4xy - 5y^2 - 3 = 0 \end{cases}$

Straight lines, $\begin{cases} 3x - 4y = 0 \\ 2x + 5y = 0 \end{cases}$

Equating each of these factors to zero, and solving with one of the given equations, we have, from the first factor,

$$(x_1, y_1) = (4, 3); \quad (x_2, y_2) = (-4, -3);$$

from the second factor,

$$(x_3, y_3) = (\tfrac{1}{2}\sqrt{-5},\ -\tfrac{1}{5}\sqrt{-5}); \quad (x_4, y_4) = (-\tfrac{1}{2}\sqrt{-5},\ \tfrac{1}{5}\sqrt{-5}).$$

Hence two solutions are real and two imaginary. The figure shows the graphs of the given equations and of the factors of the auxiliary equation.

To solve by the second method, transpose the constant term as before, then put $y = vx$.

We obtain $2x^2 - 3vx^2 = -4$; $4vx^2 - 5v^2x^2 = 3$.

Dividing, $$\frac{2x^2 - 3vx^2}{4vx^2 - 5v^2x^2} = -\frac{4}{3} \quad \text{or} \quad \frac{2 - 3v}{4v - 5v^2} = -\frac{4}{3}.$$

Clearing, etc., $20v^2 - 7v - 6 = 0$.

Hence $v = \frac{3}{4}$ or $-\frac{2}{5}$.

Therefore $y = \frac{3}{4}x$ or $y = -\frac{2}{5}x$.

(These are the linear factors of the auxiliary equation found above.)

Substituting these values of y in either of the given equations, we find x as before.

Example 2.

$$9x^2 + xy + 2y^2 = 60,$$
$$8x^2 - 3xy - y^2 = 40.$$

The auxiliary equation is

$$6x^2 - 11xy - 7y^2 = 0,$$

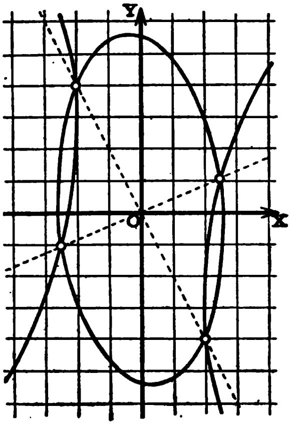
or $(2x + y)(3x - 7y) = 0$.

Solving each factor with one of the given equations we obtain

$(x_1, y_1) = (2, -4)$; $(x_2, y_2) = (-2, 4)$;

$$(x_3, y_3) = \left(\frac{7}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}\right);$$

$$(x_4, y_4) = \left(-\frac{7}{2\sqrt{2}}, -\frac{3}{2\sqrt{2}}\right).$$



Ellipse, $9x^2 + xy + 2y^2 = 60$
Hyperbola, $8x^2 - 3xy - y^2 = 40$
Straight lines, $(2x + y)(3x - 7y) = 0$

The graphs are given in the figure.

Exercises. Solve for x and y:

1. $\begin{cases} x^2 + y^2 = 9, \\ x^2 - xy = 10. \end{cases}$
2. $\begin{cases} x^2 - y^2 = 1, \\ x^2 - xy + y^2 = 1. \end{cases}$
3. $\begin{cases} 4x^2 - 9y^2 = 36, \\ y^2 + xy = -4. \end{cases}$
4. $\begin{cases} x^2 + 2xy = 2. \\ 2xy - y^2 = 6, \end{cases}$
5. $\begin{cases} x^2 + xy + y^2 = 3, \\ 2x^2 - 3y^2 = 6. \end{cases}$
6. $\begin{cases} 2x^2 + xy - 3y^2 = 2, \\ x^2 - xy + 2y^2 = 1. \end{cases}$

105. Case 3. The given equations are of the forms

$$ax^2 + by^2 = c,$$
$$a'x^2 + b'y^2 = c'.$$

Solution. Consider x^2 and y^2 as the unknowns, and solve by the method of linear equations.

Graphically, we have two conics in standard form. The four solutions may all be real, or equal or imaginary in pairs.

Example.

$$x^2 - 4y^2 = 4,$$
$$9x^2 + 16y^2 = 144.$$

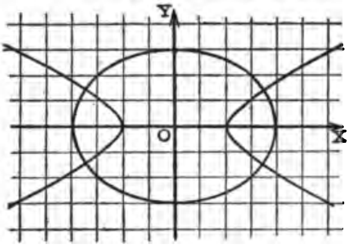
By elimination we obtain,

$$x^2 = \tfrac{160}{13}; \quad y^2 = \tfrac{27}{13}.$$

Hence

$$x = \pm 4\sqrt{\tfrac{10}{13}};$$
$$y = \pm 3\sqrt{\tfrac{3}{13}}.$$

Taking either value of x with either value of y, we obtain the four solutions. The approximate values may be scaled off from the figure.



Hyperbola, $x^2 - 4y^2 = 4$
Ellipse, $9x^2 + 16y^2 = 144$

Exercises. Solve for x and y, and check graphically:

1. $\begin{cases} x^2 + y^2 = 4, \\ x^2 - y^2 = 2. \end{cases}$

2. $\begin{cases} x^2 - y^2 = 1, \\ x^2 + 4y^2 = 4. \end{cases}$

3. $\begin{cases} 2x^2 + 5y^2 = 10, \\ 4x^2 + y^2 = 4. \end{cases}$

4. $\begin{cases} x^2 + y^2 = 9, \\ 4x^2 + 9y^2 = 36. \end{cases}$

5. $\begin{cases} 4x^2 + 5y^2 = 20, \\ x^2 - y^2 = 9. \end{cases}$

6. $\begin{cases} x^2 + y^2 = 1, \\ x^2 + y^2 = 4. \end{cases}$

106. Case 4. Symmetric and Skew-Symmetric Equations. — A *symmetric equation* is one which remains unchanged when the variables are interchanged.

A *skew-symmetric equation* is one whose variable terms all change sign when the variables are interchanged. Thus

$$x^3 + y^3 + x + y = 0, \qquad x^3 - y^3 + 2x - 2y = 1$$

are symmetric and skew-symmetric respectively.

Solution. Given two such equations, put

$$x = u + v \quad \text{and} \quad y = u - v;$$

solve the resulting equations for u and v.

Note. Equations of higher degree than the second may often be solved by this method.

Example.

$$x^4 + y^4 - x^2y^2 = 9,$$
$$x^2 + y^2 - xy = 3.$$

Let $x = u + v$ and $y = u - v$

Substituting and reducing:

$$u^4 + 14\,u^2v^2 + v^4 = 9,$$
$$u^2 + 3\,v^2 = 3.$$

Let $u^2 = s \quad \text{and} \quad v^2 = t.$

Then $s^2 + 14\,st + t^2 = 9,$
$$s + 3\,t = 3.$$

Solving: $(s, t) = (3, 0) \text{ or } (\tfrac{3}{4}, \tfrac{3}{4}).$

(If s and t be considered as the coördinates of a point, the equations in s and t represent a hyperbola and a straight line respectively.)

Since $u = \pm\sqrt{s} \quad \text{and} \quad v = \pm\sqrt{t},$

we have $(u, v) = (\pm\sqrt{3}, 0) \quad \text{or} \quad \left(\pm\frac{\sqrt{3}}{2},\ \pm\frac{\sqrt{3}}{2}\right),$

where the signs are to be taken in all possible ways.

Then

$$x = u + v = \sqrt{3},\ -\sqrt{3},\ \sqrt{3},\ -\sqrt{3},\ 0,\ 0;$$
$$y = u - v = \sqrt{3},\ -\sqrt{3},\ 0,\ 0,\ \sqrt{3},\ -\sqrt{3}.$$

Here corresponding values of x and y appear in the same vertical line.

107. Case 5. Symmetric Solution. — This method of solution is applicable to certain forms of symmetric equations, and may be illustrated by some simple examples.

Example 1. $x + y = 5,$
$$xy = 4.$$

Squaring the first equation: $x^2 + 2\,xy + y^2 = 25.$
Subtracting four times the second: $x^2 - 2\,xy + y^2 = 9.$

Hence $x - y = \pm 3.$
But $x + y = 5.$

$$\therefore \quad x = 4 \text{ or } 1; \quad y = 1 \text{ or } 4.$$

Example 2. (1) $x^2 + xy + y^2 = 6.$
(2) $x^2 - xy + y^2 = 10.$

Subtract (2) from (1): $2\,xy = -4, \quad \text{or} \quad xy = -2.$

Add $xy = -2$ to (1): $x^2 + 2\,xy + y^2 = 4, \quad \text{or} \quad x + y = \pm 2.$

Subtract $3\,xy = -6$ from (1): $x^2 - 2\,xy + y^2 = 12, \quad \text{or} \quad x - y = \pm 2\sqrt{3}.$

Hence $x = \pm 1 \pm \sqrt{3} \quad \text{and} \quad y = \pm 1 \mp \sqrt{3}.$

Simultaneous values of x and y are then obtained by taking the same combination of signs in these two results.

108. Special Methods. — These methods depend on reducing the given equations, which may be of higher degree than the second, to one of the cases already discussed.

1. **By Substitution.** — This method has already been illustrated in several cases; in (**104**) we made the substitution $y = vx$, in (**106**) we put $x = u + v$ and $y = u - v$, and in the same example we put $u^2 = s$ and $v^2 = t$. We shall give two more simple illustrations.

Example 1.

$$\frac{1}{x} + \frac{1}{y} = 2,$$
$$\frac{1}{xy} = -15.$$

If we let $\frac{1}{x} = s$ and $\frac{1}{y} = t$ we obtain,

$$s + t = 2,$$
$$st = -15.$$

These may be solved by the method of (**107**).

Example 2.

$$x^2 + y^2 + x^2y^2 + 2\,xy = 4,$$
$$x^2y^2 - 2\,xy = 0.$$

Let $x + y = s$ and $xy = t$. Then

$$s^2 + t^2 = 4,$$
$$t^2 - 2\,t = 0.$$

The last two equations are readily solved, and give

$$s = +2;\ -2;\ 0.$$
$$t = 0;\ 0;\ 2.$$

The values of x and y may now be found by solving the pairs of equations,

$$\begin{cases} x + y = 2, \\ xy = 0. \end{cases} \qquad \begin{cases} x + y = -2, \\ xy = 0. \end{cases} \qquad \begin{cases} x + y = 0, \\ xy = 2. \end{cases}$$

2. **By modifying or combining the given equations** so as to obtain simpler forms. In particular, a common factor may sometimes be removed by division.

Example 1.

$$(1) \qquad x^2 - xy = 18\,y,$$
$$(2) \qquad xy - y^2 = 2\,x.$$

Dividing (1) by (2), we have

$$\frac{x}{y} = 9\frac{y}{x} \quad \text{or} \quad \left(\frac{x}{y}\right)^2 = 9 \quad \text{or} \quad x = \pm 3\,y.$$

Substituting each of these values of x in either of the given equations, we can solve for y and so complete the solution.

Example 2.

$$\begin{matrix} (1) \\ (2) \end{matrix} \qquad \begin{cases} x^2y - x = 1, \\ x^4y^2 - x^2 = 3; \end{cases} \quad \text{or} \quad \begin{cases} x\,(xy - 1) = 1, \\ x^2\,(x^2y^2 - 1) = 3. \end{cases}$$

Divide (2) by (1):
$$x\,(xy + 1) = 3.$$

Divide this equation by (1): $\dfrac{xy+1}{xy-1} = 3.$

Hence $xy = 2.$

Then from (2), $x^2(4-1) = 3,$ or $x^2 = 1,$ or $x = \pm 1.$

But from (1), $x(2-1) = 1,$ or $x = 1.$

In this case the value $x = -1$ must be discarded.

Hence the only solution is $x = 1,\ y = 2.$

Example 3.

(1) $x^4 + y^4 = 1,$

(2) $x - y = 1.$

Raise (2) to the fourth power and subtract from (1):

(3) $4x^3y - 6x^2y^2 + 4xy^3 = 0.$

Square (2) and multiply the result by $4xy$:

(4) $4x^3y - 8x^2y^2 + 4xy^3 = 4xy.$

Subtract (4) from (3):

$$2x^2y^2 = -4xy, \quad \text{or} \quad x^2y^2 + 2xy = 0.$$

Hence $xy = 0,$ or $xy = -2.$

Solving each of the last two equations with (2) we have

$$(x, y) = (1, 0), (0, -1), \left(\frac{1+\sqrt{-7}}{2}, \frac{-1+\sqrt{-7}}{2}\right), \left(\frac{1-\sqrt{-7}}{2}, \frac{-1-\sqrt{-7}}{2}\right).$$

All four solutions also satisfy equation (1).

109. Summary of Methods for Solving Simultaneous Equations. — [Let the given equations be numbered (1) and (2)].

(a) Equation (1) linear, (2) quadratic.

Rule: Substitute from (1) in (2). *Graph,* straight line and conic.

(b) Equations (1) and (2) both quadratic.

Case 1. *Equation* (1) *is factorable.*

Rule: Put each factor separately equal to zero and solve with (2) as in (a). *Graph,* two straight lines and a conic.

Rule for factoring: Solve for y in terms of x (or x in terms of y); the quantity under the radical must be a perfect square.

Case 2. (1) $Ax^2 + Bxy + Cy^2 = D$; (2) $A'x^2 + B'xy + C'y^2 = D'$. Form the auxiliary equation, (1) $\times D'$ − (2) $\times D = 0$. Factor this and solve as in Case 1.

Second Method: Put $y = vx$ in (1) and (2) and divide results. *Graph,* two conics, centers at origin (except in case of parabola).

Case 3. (1) $Ax^2 + By^2 = C$; (2) $A'x^2 + B'y^2 = C'$.

Solve as linear equations for x^2 and y^2.

Graph, two conics in standard position.

Case 4. *Symmetric Equations.*
Put $x = u + v$ and $y = u - v$.
Applicable to equations of higher degree.

Case 5. *Symmetric Solution* of certain symmetric equations.

(c) Special Methods.

110. Exercises.

1. $x^2 + y^2 = 832.$
$x^2 - y^2 = 320.$

2. $x^2 - y^2 = 56.$
$x^2 + y^2 = 72.$

3. $12x^2 - y^2 = 59.$
$8x^2 + 3y^2 = 98.$

4. $x + y = 6.$
$4xy = 35.$

5. $x + 2y = 2.$
$xy = -2.$

6. $x^2 + y^2 = 74.$
$x - y = 12.$

7. $6x - 6y = 7.$
$3xy = -1.$

8. $\frac{x + 2y}{x - 2y} = 12.$
$x^2 - 4y^2 = 48.$

9. $5x^2 + 2y^2 = 373.$
$2x + 5y = 54.$

10. $x^2 + y^2 = -10.$
$x - y = 2.$

11. $4x^2 - y^2 = 480.$
$2x + y = 40.$

12. $9x^2 - y^2 = -\frac{9}{4}.$
$3x + y = \frac{9}{2}.$

13. $x^2 - xy = 30.$
$xy - y^2 = -6.$

14. $x^2 + y^2 = 109.$
$xy - y^2 = 21.$

15. $25x + 2y = 145.$
$xy = -105.$

16. $xy = 400.$
$x = y.$

17. $x^2 - 3y^2 = -83.$
$3x + 4y = 52.$

18. $3x^2 - y^2 = 83.$
$x - y = 15.$

19. $xy + x = 10.$
$2xy - y = 12.$

20. $x + 3y = 20.$
$3xy - 2y^2 = 76.$

21. $5x^2 - y^2 = 109.$
$14x - 5y = 50.$

22. $x - xy - y = 47.$
$x - y = 12.$

23. $x^2 - xy + y^2 = 217.$
$x - y = 17.$

24. $\frac{1}{x} + \frac{1}{y} = \frac{1}{2}.$
$\frac{1}{x^2} + \frac{1}{y^2} = \frac{25}{36}.$

25. $\frac{1}{x} + \frac{1}{y} = \frac{7}{6}.$
$\frac{1}{xy} = \frac{1}{3}.$

26. $x^2 + xy + y^2 = 37.$
$x^2 - y^2 = 40.$

27. $x^2 + xy + y^2 = 16.$
$x^2 - xy + y^2 = 8.$

28. $x^2 + xy + y^2 = 7.$
$x + y + xy = 5.$

29. $x^2 + y^2 = 5(x + y).$
$xy = 3.$

30. $x^3 + y^3 = 72.$
$xy = 16.$

31. $x^2 - y^2 = 4.$
$x^2 + xy + 2x + 2y = 0.$

32. $2x^2 + x + 3xy = 2y^2 - 2y.$
$x^2 + 2y = x + 4y^2.$

33. $u^2 + v^2 - uv = 268.$
$u - v = 18.$

34. $p^2 + pq + q^2 = 109.$
$p^2 - pq + q^2 = 39.$

35. $4r^2 + s^2 - 2rs = 100.$
$2r - s = 10.$

36. $r^2 + 4s^2 + 2rs = 84.$
$r + 2s = 2.$

37. $u + v + u^2 + v^2 = 162.$
$u - v + u^2 - v^2 = -102.$

38. $2p + 3q + 4p^2 + 9q^2 = \frac{18}{8}.$
$3q - 2p + 9q^2 - 4p^2 = -1.$

39. $x^2 + y^2 + x - y = 18.$
$xy = -6.$

40. $h^2 + k^2 + 3h - 3k = 40.$
$hk = 15.$

41. $x^2 + y^2 + x + y = 168.$
$\sqrt{xy} = 6.$

42. $m^2 + n^2 - 2m = 145.$
$2mn - 2n = 110.$

43. $9u^2 + 4v^2 + 3u + 2v = 3042.$
$\sqrt{2uv} = 12.$

44. $r^3 - s^3 = 163.$
$r - s = 4.$

45. $p^3 + q^3 = 152.$
$p + q = 8.$

46. $x^4 - y^4 = 175.$
$x^2 + y^2 = 25.$

47. $p^4 + q^4 = 337.$
$p + q = 7.$

48. $m^3 - n^3 = 152.$
$m^2 + mn + n^2 = 19.$

49. $u^4 + v^4 = 2657.$
$u + v = 11.$

50. $x^3 - y^3 = 280.$
$x^2 + xy + y^2 = 28.$

51. $u^2 + 12v^2 = 28.$
$7u^2 - 20uv = 72.$

52. $p^3 - q^3 = 152.$
$pq^2 - p^2q = 120.$

53. $x^3 + y^3 = 2608.$
$xy^2 + x^2y = -560.$

54. $s^3 - t^3 = 855.$
$st^2 - s^2t = 840.$

55. $m^3 + n^3 = 602.$
$mn(m + n) = -198.$

56. $u^2v^4 + v^2 = 17.$
$uv^2 + v = 5.$

57. $x^{\frac{3}{2}} + y^{\frac{3}{2}} = 35.$
$x^{\frac{1}{2}} + y^{\frac{1}{2}} = 5.$

58. $x^2y^2 - 18xy + 72 = 0.$
$6x^2 - 17xy + 12y^2 = 0.$

59. $x^4 + x^2y^2 + y^4 = 91.$
$x^2 - xy + y^2 = 7.$

60. $x^3 - y^3 = 7(x^2 - y^2).$
$x^2 + y^2 = 10(x + y).$

61. $s^6 + t^6 = 4.$
$s^2 + t^2 = 2.$

62. $x^2 + y^2 = a.$
$x^2 - y^2 = b.$

63. $x - y = m.$
$xy = n^2.$

64. $p^2 + q^2 = a^2.$
$p + q = b.$

65. $\sqrt{u} + \sqrt{v} = a.$
$u + v = b^2.$

66. $x^2 + y^2 = a(x - y).$
$x^3 + y^3 = b(x + y).$

67. $ax - by = m.$
$a^3x^3 - b^3y^3 = nxy.$

68. $b(x + y) = a(x - y).$
$x^2 + y^2 = m^2.$

69. $x^4 + y^4 = -8.$
$x - y = 2.$

70. $p^4 + q^4 = -9.$
$p - q = 3.$

71. $x^{-3} + y^{-3} = 35.$
$x + y = 5xy.$

72. $\sqrt[3]{x^2} + \sqrt[3]{y^2} = 5.$
$x^2 + y^2 = 65.$

73. $\frac{1}{x} - \frac{1}{y} = \frac{1}{a}.$
$\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{b^2}.$

74. $u^2 + uv = m.$
$v^2 + uv = n.$

75. $x^3 + xy^2 = p.$
$y^3 + x^2y = q.$

76. $m^3 - n^3 = a(m - n).$
$m^3 + n^3 = b(m + n).$

77. $r^3 + s^5 = 3368.$
$r + s = 8.$

78. $x(x + y - z) = 1.$
$y(x + y - z) = 2.$
$z(x + y - z) = 3.$

79. $x + y + z = 2.$
$xy = 3.$
$xyz = 6.$

80. $xy = 8z.$
$xz = 18y.$
$yz = \frac{9}{2}x.$

81. $xy + x = 1.$
$yz + y = -1.$
$xz + z = 3.$

82. $x(x + y - z) = a.$
$y(x + y - z) = b.$
$z(x + y - z) = c.$

83. $x + y + z = p.$
$xy = q.$
$xyz = r.$

84. $x(x + y + z) = a^2.$
$y(x + y + z) = b^2.$
$z(x + y + z) = c^2.$

85. $(x + y)(x + z) = 4.$
$(x + y)(y + z) = 1.$
$(x + z)(y + z) = 16.$

86. $(x + y)(x + z) = a.$
$(x + y)(y + z) = b.$
$(x + z)(y + z) = c.$

87. $xy + x = a.$
$yz + y = b.$
$xz + z = c.$

88. $x^2 + y^2 = 17z.$
$3(x + y) = 5z.$
$x - y = z.$

89. $x^2 + y^2 + z^2 = \frac{181}{144}.$
$y^2 + x = \frac{59}{48}.$
$z^2 + x = \frac{11}{12}.$

Problems.

1. The hypothenuse of a right triangle is 15 ft. long. Find the other sides, if their ratio is 3 : 4.

2. The product of two numbers is 384, and their quotient $\frac{3}{2}$. Find the numbers.

3. Find two factors of 1728 whose sum is 96.

4. The sum of two numbers is 29. Three times their product exceeds the sum of their squares by 179. What are the numbers?

5. The product of two numbers increased by the first is 735, increased by the second is 720. What are the numbers?

6. The product of two numbers times their sum is 1890, times their difference 810. What are the numbers?

7. The sum of the squares of two numbers plus the sum of the numbers is 686. The difference of the squares plus the difference of the numbers is 74. What are the numbers?

8. The diagonal of a rectangle is 89 ft. long. If each side were 3 ft. shorter, the diagonal would be 4 ft. shorter. Find the sides.

9. The diagonal of a rectangle is 65 ft. long. If the shorter side were decreased by 17 ft. and the longer increased by 7 ft., the diagonal would be unchanged. Find the sides.

10. The diagonal of a rectangle is 85 ft. long. If each side be increased 2 ft. in length, the area is increased by 230 sq. ft. Find the sides.

11. The floor area of two square rooms is 890 sq. ft., and one room is 4 ft. larger each way than the other. Find the dimensions of each room.

12. For 60 yards of cloth B pays two dollars more than A pays for 45 yards. B receives one yard more for two dollars than does A. How much does each pay per yard?

13. Two bodies moving around the circumference of a circle of length 1260 ft. pass each other every 157.5 seconds. The first body makes the circuit in 10 seconds less than the second. Find the speed of each body.

14. The amount of a capital plus interest for one year is $22,781. If the capital were $200 larger and the rate of interest $\frac{1}{4}$% larger, the amount in one year would be $23,045. Find the capital and rate of interest.

15. A and B agree to do a piece of work in 6 days for $45. To finish on time, they hire C during the last two days, and consequently B gets $2 less pay. If A could have done the work alone in 12 days, how long would it take B and C, each working alone, to do it?

16. Find the dimensions of a rectangle whose diagonal is 13 and area 60. Solve when the diagonal is d and the area is a.

17. A rectangle whose area is 192 is inscribed in a circle whose area is $100\,\pi$. Find the dimensions of the rectangle. Generalize this problem.

18. What number of two digits is 5 greater than twice the product of its digits and 4 less than the sum of their squares?

19. A fraction is doubled by adding 6 to its numerator and taking 2 from its denominator. If the numerator be increased and the denominator decreased by 3, the fraction is changed to its reciprocal. What is the fraction?

20. A and B start at the same time from two points 221 miles apart and travel towards each other. A goes 10 miles a day. B goes as many miles a day as the number of days until they meet diminished by 6. How far did each one travel?

21. The fore wheel of a wagon makes 1000 revolutions more than the hind wheel in going a distance of 7500 yards. Had the circumference of each wheel been one yard more, the difference between the number of revolutions would nave been 625. Find the circumference of each wheel.

22. Find two numbers such that their sum shall be equal to 28, and the sum of their cubes divided by the sum of their squares equal to 1456.

23. Two points, A on the x-axis 270 ft. from the origin and B on the y-axis 189 ft. from the origin, move toward the origin. After 10 seconds the distance between them is 169 ft., and after 14 seconds, 109 ft. Find the speed of each point.

111. Exponential Equations. — An exponential equation is one in which the unknown appears in the exponent. Thus:

$$\sqrt{a^x} = a^{2x-1};\quad (m^{x+1})^x = m^{-2x-2};\quad a^{x+1} = b^{2x-1}.$$

Exponential equations of the above forms may be solved by reduction to ordinary equations by use of the principle that

if $$a^u = a^v, \text{ then } u = v,$$

or more generally

if $$a^u = b^v, \quad \text{then} \quad u \log a = v \log b.$$

Example 1. $$\sqrt{a^x} = a^{2x-1}.$$

This may be written $a^{\frac{x}{2}} = a^{2x-1}$.

$$\therefore\quad \frac{x}{2} = 2x - 1 \quad \text{or} \quad x = \frac{2}{3}.$$

Example 2.

$$(m^{x+1})^x = m^{-2x-2}.$$

$$m^{x^2+x} = m^{-2x-2}.$$

$$\therefore\quad x^2 + x = -2x - 2 \quad \text{or} \quad x^2 + 3x + 2 = 0.$$

Hence $$x = -2 \quad \text{or} \quad -1.$$

Example 3. $2^{x+1} = 3^{2x-1}.$

Taking logarithms: $(x + 1) \log 2 = (2x - 1) \log 3.$

$\therefore \quad x (\log 2 - 2 \log 3) = -\log 2 - \log 3.$

or
$$x = \frac{\log 2 + \log 3}{2 \log 3 - \log 2} = \frac{\log 6}{\log \frac{9}{2}}.$$

Using common logarithms to four places,
$$x = \frac{0.7781 +}{0.6532 +} = 1.1912 +.$$

112. Exercises. — Solve:

1. $2^x = 8.$
2. $2^x = \frac{1}{8}.$
3. $4^x = \frac{1}{64}.$
4. $(\frac{1}{4})^x = 2^{12}.$
5. $(\frac{1}{2})^x = 1.$
6. $(\frac{1}{125})^x = 25^3.$
7. $10^{-x} = 1000.$
8. $1000^x = 100.$
9. $3^{x+2} = 3^3.$
10. $\sqrt[5]{a^{x-2}} = a^{x-8}.$
11. $\sqrt[5]{p^{2x+8}} = p^0.$
12. $4^{2x-1} = 2^{6x+8}.$
13. $\sqrt[3]{m^x} = \sqrt{m^{3x+2}}.$
14. $\dfrac{1}{a^{5x}} = \sqrt[3]{a^{6-13x}}.$
15. $\sqrt[2x+1]{a^4} = \sqrt[8x-1]{a^5}.$
16. $3^x = \sqrt[3]{5}.$
17. $3^{x+1} = 5^{2x}.$
18. $4^{2x-3} = 7^{x-1}.$
19. $a^{3x+2} = b^{2x-1}.$
20. $3^{x^2-x-6} = 1.$
21. $8^{x^2+2x} = 512.$
22. $5^{\frac{x^3-8}{x-2}} = 25^{\frac{x^2-1}{2(x+1)}}.$
23. $(a^{x-2})^{3x+4} = a^{x(3x+\frac{1}{2})}.$
24. $(b^{x+3})^{3x-1} = b^{3x(x+1)}.$
25. $\sqrt[3]{e^{2-x}}\,\sqrt[4]{e^{4-x}}\,\sqrt[6]{e^{5x-1}} = 1.$

CHAPTER VI

RATIO, PROPORTION, VARIATION

113. Definitions.—The **ratio** of two quantities is their indicated quotient.

Thus the ratio of a to b is $\frac{a}{b}$, or as it is usually written, $a : b$.

The numerator of the fraction, or the first term of the ratio, is called the **antecedent,** the other term the **consequent.**

The ratio $b : a$ is called the **inverse** of the ratio $a : b$.

Two ratios are equal when the fractions representing them are equal.

$$\text{Since} \qquad \frac{a}{b} = \frac{ma}{mb}, \qquad \therefore \qquad a : b = ma : mb.$$

Hence *both terms of a ratio may be multiplied by the same (or equal) quantities without altering the value of the ratio.*

Similarly, if $m \neq n$, then $a : b \neq ma : nb$.

Hence *if the terms of a ratio be multiplied by unequal quantities, the value of the ratio is changed.*

A *proportion* is an equality of two ratios. Four numbers are in proportion when the ratio of two of them equals the ratio of the other two.

Four numbers a, b, c, d are in proportion if $a : b = c : d$ (often written $a : b :: c : d$). Here a and d are called the *extremes* and b and c the *means.* Also, d is called a *fourth proportional* to a, b, c.

The numbers $a, b, c, d, e, \cdots$ are in *continued proportion* if

$$a : b = b : c = c : d = d : e \cdots.$$

When three numbers a, b, c are in continued proportion, so that $a : b = b : c$, then b is called *a mean proportional* between a and c.

Since $\frac{a}{b} = \frac{b}{c}$ or $ac = b^2$ we have $b = \pm\sqrt{ac}$. Also, c is called the third proportional to a and b.

114. Laws of Proportion.

1. In a proportion, the product of the means equals the product of the extremes.

2. If two products, each containing two factors, are equal, either pair of factors may be taken as the means, the other as the extremes of a proportion.

When four numbers are in proportion so that $a : b = c : d$, then they are in proportion

3. by inversion, or $b : a = d : c$;

4. by alternation, or $a : c = b : d$;

5. by composition, or $a + b : b = c + d : d$;

$$\left(\text{if } \frac{a}{b} = \frac{c}{d}, \quad \text{then} \quad \frac{a}{b} + 1 = \frac{c}{d} + 1, \quad \text{or} \quad \frac{a+b}{b} = \frac{c+d}{d}\right);$$

6. by division, or $a - b : b = c - d : d$;

7. by composition and division, or $a + b : a - b = c + d : c - d$.

8. Like powers (or roots) of the terms of a proportion are in proportion, i.e.,

$$\text{if} \qquad a : b = c : d, \qquad \text{then} \qquad a^n : b^n = c^n : d^n.$$

$$\left(\text{For if } \frac{a}{b} = \frac{c}{d}, \quad \text{then} \quad \frac{a^n}{b^n} = \frac{c^n}{d^n}.\right)$$

9. The products of the corresponding terms of any number of proportions are in proportion. Thus, if

$$a : b = c : d, \quad a' : b' = c' : d', \quad a'' : b'' = c'' : d'',$$

$$\text{then} \qquad aa'a'' : bb'b'' = cc'c'' : dd'd''.$$

$$\text{For if } \frac{a}{b} = \frac{c}{d}, \quad \frac{a'}{b'} = \frac{c'}{d'}, \quad \frac{a''}{b''} = \frac{c''}{d''}, \qquad \text{then} \quad \frac{aa'a''}{bb'b''} = \frac{cc'c''}{dd'd''}.$$

10. In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent. Thus:

$$a_1 : a_2 = b_1 : b_2 = c_1 : c_2 = a_1 + b_1 + c_1 : a_2 + b_2 + c_2.$$

$$\text{For if } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \cdots = r, \quad \text{then} \quad a_1 = a_2 r,\ b_1 = b_2 r,\ c_1 = c_2 r, \cdots.$$

$$\text{Hence} \qquad (a_1 + b_1 + c_1 + \cdots) = r\,(a_2 + b_2 + c_2 + \cdots), \text{ or}$$

$$\frac{a_1 + b_1 + c_1 + \cdots}{a_2 + b_2 + c_2 + \cdots} = r.$$

11. More generally, if the ratios $a_1 : a_2$, $b_1 : b_2$, $c_1 : c_2 \cdots$ are all equal to each other, then

(a) $$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \cdots = \frac{pa_1 + qb_1 + rc_1 + \cdots}{pa_2 + qb_2 + rc_2 + \cdots},$$

where $p, q, r, \cdots$ are any multipliers;

(b) $$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \cdots = \sqrt[n]{\frac{a_1^n + b_1^n + c_1^n + \cdots}{a_2^n + b_2^n + c_2^n + \cdots}}.$$

Exercise. Prove 11 (a) and 11 (b). For what values of $p, q, r, \cdots n$ will these reduce to 10?

115. Exercises.—Solve for x, using the laws of proportion:

1. $\dfrac{x+1}{x} = \dfrac{3}{2}$.
2. $\dfrac{x}{x-2} = -\dfrac{5}{6}$.
3. $\dfrac{2x-3}{2x+3} = \dfrac{7}{6}$.
4. $3x - 2 : 3x + 2 = 3 : 4$.
5. $\dfrac{x+1}{x-1} = \dfrac{x+3}{x-4}$.
6. $\dfrac{x+a}{x} = \dfrac{b}{c}$.
7. $a - x : x = p : q$.
8. $x + m : a = x - m : b$.
9. $a - x : x - b = a : b$.
10. $\dfrac{x+p}{x-p} = \dfrac{a+x}{b+x}$.
11. $2x + 1 : 3x - 1 = 4x - 3 : 6x + 2$.

From the proportion $a : b = c : d$ build up the following:

12. $4a + 5b : 3a + 2b = 4c + 5d : 3c + 2d$.
13. $a - 2b : -2a + b = c - 2d : -2c + d$.
14. $ma + nb : pa + qb = mc + nd : pc + qd$.
15. $3a + 2c : a - c = 3b + 2d : b - d$.
16. $\frac{1}{2}a - 4c : \frac{1}{2}b - 4d = 2a + \frac{1}{2}c : 2b + \frac{1}{2}d$.
17. $a : a + c = a + b : a + b + c + d$.
18. $a^2 + ab + b^2 : a^2 - ab + b^2 = c^2 + cd + d^2 : c^2 - cd + d^2$.
19. $a + b : c + d :: \sqrt{a^2 + b^2} : \sqrt{c^2 + d^2}$.
20. $\sqrt{a^2 + b^2} : \sqrt{c^2 + d^2} = \sqrt[3]{a^3 + b^3} : \sqrt[3]{c^3 + d^3}$.
21. $\sqrt{a^2 + b^2} : \sqrt{c^2 + d^2} = \sqrt[3]{a^3 - b^3} : \sqrt[3]{c^3 - d^3}$.

If $a : b = c : d$ and $p : q = r : s$, show that

22. $p^m a^n : r^m b^n = q^m c^n : s^m d^n$.
23. $(a + b)(p - r) : (a - b)(p + r) = (c + d)(q - s) : (c - d)(q + s)$.

Solve for x:

24. $8ab : x = bc : 1\frac{3}{4}ac$.
25. $\dfrac{a+b}{a-b} : \dfrac{a^2 - b^2}{ab} = x : \dfrac{(a-b)^2}{ab}$.
26. $\left(\dfrac{a^3 - b^3}{a - b} + ab\right) : \left(\dfrac{a^3 + b^3}{a + b} - ab\right) = (a + b)^2 : x$.

116. Variation. — A *variable* quantity is one which may be considered to assume a number of values.

A *function* of a variable is a quantity whose value depends on that of the variable.

If y be a function of a variable x [indicated by writing $y = f(x)$], then in general, as x varies, y varies with it.

Thus, the circumference of a circle depends on the radius and varies with the radius. Hence the circumference is a function of the radius [$c = f(r)$]. The functional relation is expressed by $c = 2\pi r$.

Similarly, the area of a circle depends on the radius [$A = f(r)$]. The functional relation in this case is $A = \pi r^2$.

Also, the cost of a piece of cloth depends on, or is a function of, the price per yard; the running time of a train between two stations is a function of the speed; the range of a gun is a function of the muzzle velocity.

117. Direct Variation. — *A quantity y varies directly with another quantity x when their ratio remains constant.*

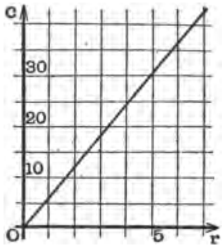
This is indicated by writing $y \propto x$ (read "y varies directly as x").

If k denote the constant value of the ratio of y to x, then **$y \propto x$ is exactly equivalent to $y = kx$.**

The constant k will be determined as soon as the value of y corresponding to a single value of x (other than $x = 0$) is known.

Graphically, the relation between y and x is represented by a straight line through the origin, the inclination of the line to the x-axis increasing with the absolute value of k. The line is completely determined by the origin ($x = 0$, $y = 0$) and one other point.

If c be the circumference and r the radius of a circle, then $c \propto r$, for $c = 2\pi r$. If we take $\pi = \frac{22}{7}$, then $c = \frac{44}{7}$ when $r = 1$. The figure gives the graph.



Horizontal scale = 5 times vertical scale

Exercise. From the figure read off to the nearest unit the lengths of circumference of circles whose radii are 1.5 in., 3.3 ft., 4.5 mm., 2.87 cm. respectively.

118. Inverse Variation. — When y varies directly as $\frac{1}{x}$, that is, $y \propto \frac{1}{x}$ or $y = \frac{k}{x}$, then y is said to *vary inversely as x.* This may also be expressed by writing $xy = k$.

Graphically, the relation between x and y is then represented by a rectangular hyperbola, whose asymptotes are the coördinate axes (**98**).

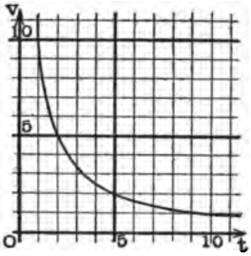
If t be the time, in hours, required by a train to run 10 miles, and v the speed in miles per hour, then

$$t = \frac{10}{v} \quad \text{or} \quad t \propto \frac{1}{v}.$$

The figure gives the graph, only positive values being considered.

Exercise 1. From the figure read off to tenths of a unit the times required to run 10 miles when $v = 4.5$, 7.8, and 10.6 miles per hour respectively.

Exercise 2. Construct a curve showing the possible dimensions of a rectangle whose area must be 16 sq. ft. Show that either dimension varies inversely as the other.



$vt = 10$

119. Joint Variation. — When a quantity varies directly as the product of two others, it is said to *vary with them jointly.*

Thus, if $z \propto xy$, or $z = kxy$, then z varies jointly as x and y.

120. Exercises.

1. Show that the area of a rectangle varies jointly as its dimensions.
2. Show that the volume of a right cylinder varies jointly as its base and altitude.
3. Same as in 2 for a right circular cone.
4. Show that the volume of a sphere varies jointly as the radius and the area of a great circle.
5. If $y \propto x$ and $x \propto z$, show that $y \propto z$.
6. If $x \propto z$ and $y \propto z$, show that $ax + by \propto z$.
7. If $x^2 \propto y$ and $z^2 \propto y$, show that $xz \propto y$.
8. If $x \propto \frac{1}{y}$ and $y \propto \frac{1}{z}$, show that $x \propto z$.
9. If x varies jointly as p and q, and y varies directly as $\frac{p}{q}$, show that p^2 varies jointly as x and y.
10. According to Boyle's law of gases, pressure times volume is constant.

Show that the pressure (p) varies inversely as the volume (v). Show graphically the relation between p and v if $v = 1$ cu. ft. when $p = 25$ lbs. per sq. in.

11. If $w = uv$, show that $w \propto u$ when v is constant, and that $w \propto v$ when u is constant.

12. The intensity of light varies inversely as the square of the distance from the source. If the sun is equivalent to 600,000 full moons in brightness, at how many times its present distance would it be of the same brightness as the full moon?

13. The squares of the periods of revolution of the planets about the sun vary as the cubes of their mean distances. The earth makes a revolution in one year at a mean distance of 93,000,000 miles. Venus makes a revolution in 225 days, Jupiter in 12 years. Find their mean distances from the sun.

14. In beams of the same width and thickness the deflection due to a central load varies jointly as the load and the cube of the length. If a beam 10 ft. long is bent ½ inch by a load of 1000 lbs., how much will a load of 500 lbs. bend a 30-ft. beam?

15. Two lights, one of which is twice as strong as the other, are 10 ft. apart. Where on the line joining them do they produce equal illumination?

Express the following statements in algebraic language:

16. The area of a circle varies as the square of the radius.

17. The surface of a sphere varies as the square of the radius, the volume as the cube of the radius.

18. The volume of a cylinder varies jointly as its base and altitude; or jointly as the square of the radius and the altitude.

19. The time of vibration of a pendulum varies as the square root of the length.

20. The force with which two bodies attract each other varies directly as the product of their masses and inversely as the square of the distance between them.

(Suggestion. In working the last five exercises, use letters suggested by the words: thus, l = length; t = time; m, m' = the masses of two bodies; and so on.)

CHAPTER VII

THE PROGRESSIONS. ANNUITIES

121. Arithmetic Progressions. — Let $a, b, c, \cdots, k, l$ be quantities such that the *difference* between any one of them and the preceding one is constant. Then the quantities are said to form an **arithmetic progression.** (We shall abbreviate this into A. P.)

The quantities $a, b, c, \cdots, k, l$ are called the *terms* of the progression, a and l the *extremes*, and $b, c, \cdots, k$ the means. The constant difference between consecutive terms is called the *common difference.*

Let a denote the first term,
l denote the last term,
d denote the common difference,
n denote the number of terms,
S denote the sum of the terms of any A. P. Then
the second term is $a + d$,
the third term is $a + 2d$,

. . . . .

the last or nth term is $a + (n - 1)d$; that is,

$$l = a + (n - 1)d. \tag{1}$$

Also

$$S = a + (a + d) + (a + 2d) + \cdots + (a + \overline{n - 1}\, d);$$
$$S = l + (l - d) + (l - 2d) + \cdots + (l - \overline{n - 1}\, d).$$

Adding,

$$2S = (a + l) + (a + l) + \cdots + (a + l) = n(a + l).$$

Hence

$$S = \frac{n}{2}(a + l). \tag{2}$$

Putting for l its value from (1),

$$S = n\left(a + \frac{n - 1}{2}d\right). \tag{2'}$$

We shall refer to the five quantities a, l, d, n, S, as the *elements* of the A. P. When any three elements are given, the other two may be found by use of the preceding formulas.

122. Problem. — *To insert m arithmetic means between two given quantities, a and l.*

Since there are 2 extremes and m means, the total number of terms is $m + 2$. Hence if d be the common difference,

$$l = a + (m + 2 - 1)\,d;$$

hence

$$d = \frac{l - a}{m + 1}.$$

Then the required means are

$$a + d,\ a + 2\,d,\ \cdots,\ a + md.$$

When $m = 1$ we have only a single mean, called **the arithmetic mean.** It equals $\frac{1}{2}(a + l)$.

123. *Examples.*

1. Find the sum of all the integers from 1 to 100 inclusive.

Here $S = 1 + 2 + 3 + \cdots + 100.$

Then $a = 1,\ l = 100,\ n = 100,$

and $S = \frac{n}{2}(a + l) = \frac{1}{2} \times 100\,(1 + 100) = 5050.$

2. How many terms of the progression 3, 0, −3, · · · are required to make the sum equal − 27.

Here $a = 3$, $d = -3$, $S = -27$; to find n.

From (2′), $-27 = n\left(3 - \frac{n-1}{2} \times 3\right)$, or $n^2 - 3n - 18 = 0.$

Hence $n = 6$ or $-3.$

Since n must be positive we discard the second value.

3. Find four numbers in A.P., such that the sum of the first and last shall be 12 and the product of the middle two 32.

Let the numbers be $a - 3\,d$, $a - d$, $a + d$, and $a + 3\,d$, with a common difference $2\,d$.

Then $a - 3\,d + a + 3\,d = 12$

and $(a - d)(a + d) = 32.$

Hence $a = 6$ and $d = \pm 2.$

Therefore the numbers are

0, 4, 8, 12, or 12, 8, 4, 0.

124. Exercises. — Find the last term and the sum of each of the following arithmetic progressions:

1. 7, 11, 15, · · · , to 13 terms;

2. 5, 8, 11, · · · , to 12 terms;

3. 2, $2\frac{1}{4}$, 3, · · · , to 25 terms;

4. 1, 1.1, 1.2, · · · , to 200 terms;

5. 63, 58, 53, · · · , to 8 terms;

6. $x, x+2\,y, x+4\,y$, · · · , to 10 terms;

7. $p, p - \frac{1}{2}q, p - q$, to 20 terms.

Find the other elements of the A.P., given that:

8. $a = 10$, $n = 14$, $S = 1050$; **16.** $n = 35$, $S = 2485$, $d = 3$;
9. $a = 3$, $n = 50$, $S = 3825$; **17.** $n = 50$, $S = 425$, $d = \frac{1}{3}$.
10. $a = -45$, $n = 31$, $S = 0$; **18.** $n = 33$, $S = -33$, $d = -\frac{3}{4}$;
11. $l = 21$, $n = 7$, $S = 105$; **19.** $S = 624$, $a = 9$, $d = 4$;
12. $l = 49$, $n = 19$, $S = 503\frac{1}{2}$; **20.** $S = 2877$, $a = 7$, $d = 3$;
13. $l = 148$, $n = 27$, $S = 2241$; **21.** $S = 623$, $d = 5$, $l = 77$;
14. $l = -143$, $n = 33$, $S = -2079$; **22.** $S = 682.5$, $d = 1.5$, $l = 45$;
15. $n = 21$, $S = 1197$, $d = 4$; **23.** $S = 95172$, $d = -7$, $l = 567$.

24. Find the sum of the first 100 odd numbers.

25. Find the sum of the first 50 multiples of 7.

26. A body starting from rest falls 16 ft. during the first second, and in every other second 32 ft. more than during the preceding. How far does the body fall in 12 seconds; how far during the 12th second?

27. According to the rate of fall in exercise 26, how long will the body take to fall 1600 ft.?

28. A body which is projected vertically upward loses 32 ft. of its initial velocity each second. If the velocity of projection is 320 ft. per second, how high will the body rise?

29. If 100 apples are laid in a straight line, 3 feet apart, how far must a person walk to carry them one at a time to a basket standing beside the first apple?

30. A regular polygon of 60 sides is drawn and each vertex is joined to every other vertex. Show that the figure contains 1770 lines.

125. Geometric Progressions. — If the numbers $a, b, c, \cdots, k, l$ are such that the *ratio* of any number to the preceding number is constant, the numbers form a **geometric progression.** (We abbreviate by writing G. P.)

The expressions "*terms*," "*means*," "*extremes*," are used here as in the case of A. P. The constant ratio of any term to the preceding is called the *ratio* of the geometric progression.

If a, l, n and S have the same meaning as in the case of the A. P., and if r denote the ratio of the G. P., the first n terms are,

$$a, ar, ar^2, ar^3, \cdots, ar^{n-1}.$$

Hence

(1) $$l = ar^{n-1}.$$

Also $$S = a + ar + ar^2 + \cdots + ar^{n-1}$$

and $$rS = ar + ar^2 + \cdots + ar^{n-1} + ar^n.$$

Therefore $$rS - S = ar^n - a,$$

or $$(r - 1)S = (r^n - 1)a.$$

Hence

(2) $$S = a\frac{r^n - 1}{r - 1} = a\frac{1 - r^n}{1 - r}.$$

Substituting from (1) in (2) we have

(2′) $$S = \frac{rl - a}{r - 1}.$$

When any three of the five elements are given, the other two may be obtained by use of two of the preceding formulas. In some cases this involves the solution of an equation of nth degree or of an exponential equation.

126. Problem. — *To insert m geometric means between two given numbers a and l.*

The total number of terms being $m + 2$, we have, if r denote the ratio,

$$l = ar^{m+2-1} \quad \text{or} \quad r = \sqrt[m+1]{\frac{l}{a}}.$$

The required geometric means are then

$$ar,\ ar^2,\ \cdots,\ ar^m.$$

When $m = 1$, the resulting single mean between a and l is $\sqrt{al}$. The square root of the product of two quantities is called their **geometric mean.**

127. *Examples.*

1. Find the sum of the first 10 terms of the G. P. 2, 2^2, 2^3, $\cdots$.

Here $a = 2,\ r = 2,\ n = 10;$ hence $S = 2\,\frac{2^{10} - 1}{2 - 1} = 2046.$

2. How many terms of the G. P. 1, 2, 4, $\cdots$ are required to make the sum 63?

Here $a = 1, r = 2, S = 63;$ to find n.

From $S = a\frac{r^n - 1}{r - 1}$ we have $63 = \frac{2^n - 1}{2 - 1}$; or, $64 = 2^n$.

Hence $n = 6$.

3. Four numbers are in geometric progression. The sum of the first and last is 18, the product of the second and third 32. Find the numbers.

Let the numbers be a, ar, ar^2 and ar^3.

Then

$$(1)\ a + ar^3 = 18; \qquad (2)\ a^2r^3 = 32.$$

Multiply (1) by a and in the result replace a^2r^3 by 32.

Then $a^2 + 32 = 18\,a;$ hence $a = 16$ or 2.

Substituting the values of a in (2) we find $r = \frac{1}{2}$ or 2. Hence the numbers are

16, 8, 4, 2; or 2, 4, 8, 16.

(We disregard the imaginary values of r.)

128. Exercises. — Find the last term and the sum of the terms of the following geometric progressions:

1. 4, 8, 16, $\cdots$, to 7 terms.

2. 2, 6, 18, $\cdots$, to 9 terms.

3. 1, 4, 16, $\cdots$, to 7 terms.

4. 9, 3, 1, $\cdots$, to 11 terms.

5. 1, $\frac{1}{4}$, $\frac{1}{16}$, $\cdots$, to 10 terms.

6. 8, 2, $\frac{1}{2}$, to 20 terms.

7. a, $a(1+x)$, $a(1+x)^2$, $\cdots$ to 8 terms.

8. m^3, mn, $m^{-1}n^2$, $\cdots$, to 9 terms.

9. Insert 3 geometric means between 8 and 10368.

10. Insert 5 geometric means between 2 and 31250.

11. Insert 5 geometric means between 36 and $\frac{4}{81}$.

12. Insert 6 geometric means between 3 and 49152.

13. Insert 4 geometric means between 48 and $\frac{3}{64}$.

14. Insert 5 geometric means between 81 and $\frac{256}{81}$.

Calculate the unknown elements, given:

15. $l=128$, $r=2$, $n=7$.

16. $l=78125$, $r=5$, $n=8$.

17. $l=\frac{2}{27}$, $r=\frac{1}{3}$, $n=5$.

18. $a=9$, $l=2304$, $r=2$.

19. $a=2$, $l=64$, $r=2$.

20. $a=3$, $l=192\sqrt{2}$ $r=\sqrt{2}$,

21. $a=2$, $l=1458$, $S=2186$.

22. $a=1$, $l=2401$, $S=2801$.

23. $a=10$, $l=\frac{5}{16}$, $S=19\frac{11}{16}$.

24. $a=3125$, $l=5$, $S=3905$.

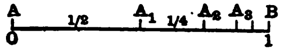
25. $a=3$, $r=3$, $S=29523$.

26. $a=8$, $r=2$, $S=4088$.

27. $r=2$, $n=7$, $S=635$.

28. $l=1296$, $r=6$, $S=1555$.

129. Infinite Geometric Progressions. — Consider a line segment AB of unit length, and bisect it at A_1, then bisect A_1B at A_2, A_2B at A_3 and so on (figure).



The points of bisection $A_1, A_2, A_3, \cdots$ continually approach B and the sum of the segments $AA_1 + A_1A_2 + A_2A_3 + \cdots$ approaches AB or 1. But the sum of these segments is represented numerically by the series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots, \quad \text{or} \quad \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots,$$

and hence by taking n large enough we can make the sum

$$S_n=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}$$

differ from 1 by as little as we please. Hence we take

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots \text{ to infinity} = 1.$$

The sum S_n above is a geometric progression with $r=\frac{1}{2}$ and $a=\frac{1}{2}$. Its sum to n terms is therefore

$$S_n = \frac{1}{2}\frac{(\frac{1}{2})^n - 1}{\frac{1}{2} - 1}.$$

As n increases, $(\frac{1}{2})^n$ approaches 0, and S_n approaches the value $\frac{1}{2}\frac{0 - 1}{\frac{1}{2} - 1} = 1$, as found above.

A geometric progression in which the number of terms increases without limit is called an *infinite geometric progression.*

For the sum of n terms of any G. P. we have

$$S_n = a\frac{r^n - 1}{r - 1} = a\frac{1 - r^n}{1 - r}.$$

If now $|r| < 1$, then r^n approaches 0 when n approaches ∞, and the formula for the sum of an infinite G. P. is

$$S = \frac{a}{1 - r}, \quad \text{provided } |r| < 1.$$

(When $r = 1$, or when $r > 1$, S is infinite.)

Example 1. A ball is thrown vertically upward to a height of 60 ft. On striking the ground it always rebounds to one-third the height from which it fell. How far will it travel?

The distance covered during the first rise and fall is 120 ft., during the second rise and fall, $\frac{1}{3} \times 120$ ft., during the third, $\frac{1}{3^2} \times 120$ ft., and so on indefinitely. We have an infinite G.P., with $a = 120$ and $r = \frac{1}{3}$. Hence the total distance will be

$$S = \frac{120}{1 - \frac{1}{3}} = 180 \text{ ft.}$$

Example 2. Express the repeating decimal 0.131313 · · · as an ordinary fraction.

$$\begin{aligned} 0.131313 \cdots &= \frac{13}{100} + \frac{13}{100^2} + \frac{13}{100^3} + \cdots \\ &= \frac{13}{100}\left[1 + \frac{1}{100} + \frac{1}{100^2} + \cdots\right] \\ &= \frac{13}{100} \cdot \frac{1}{1 - \frac{1}{100}} = \frac{13}{100} \cdot \frac{100}{99} = \frac{13}{99}. \end{aligned}$$

130. Exercises.—Sum the following infinite geometric progressions:

1. $8, 2, \frac{1}{2}, \cdots.$
2. $1, \frac{1}{4}, \frac{1}{16}, \cdots.$
3. $5, 3, \frac{9}{5}, \cdots.$
4. $2, \frac{2}{7}, \frac{2}{49}, \cdots.$
5. $1, -\frac{1}{2}, +\frac{1}{4}, -\frac{1}{8}, \cdots.$
6. $3, -1, \frac{1}{3}, -\frac{1}{9}, \cdots.$

Express the following repeating decimals as fractions.

7. 0.333 · · · . **8.** 0.222 · · · . **9.** 0.161616 · · · .
10. 0.353535 · · · . **11.** 0.123123123 · · · .

12. If in the example worked above the ball requires 4 seconds for the first rise and fall, and half as much time for any subsequent rise and fall as for the preceding, how long before the ball will come to rest?

13. How far has the ball in the above example traveled at the 10th rebound?

131. Harmonic Progressions. — If the numbers $a, b, c, \cdots, k, l$ are such that their reciprocals form an arithmetic progression, they are said to be in harmonic progression (abbreviated to H. P.).

Problems relating to harmonic progressions are solved by reduction to A. P.

If a, b, c form a H. P., then b is called the harmonic mean between a and c. Let the student show that we then have

$$b = \frac{2\,ac}{a+c}.$$

132. Exercises.

1. In an A. P. the sum of the 9th and 12th terms is 40; the difference between the squares of the 15th and 11th terms is 400. Find a and d.

2. In an A. P. of 10 terms, the sum of the terms is 65 and the sum of their squares 1165. Find a and d.

3. In an A. P. of 20 terms, the sum of the 3rd and 12th terms is 30, the product of the two middle terms is 725. Find a and d.

4. In an A. P. of 14 terms, the product of the first and the last is 276 and the product of the middle two is 1326. Find a and d.

5. Find four numbers in A. P. such that their product is 840 and their sum 11.

6. Find four numbers in A. P. such that their product is h and the sum of their squares is k.

7. Find five numbers in A. P. such that their product is a, their sum $5\,b$.

8. Determine the sides of a right triangle if they form an A. P. with a common difference 2.

9. Find the angles of a triangle if they form an A. P. with $d = 5°$.

10. Between every pair of consecutive terms of the G. P. 1, 2, 4, 8, · · · insert a new term so that the result is again a G. P.

11. As in exercise 10 for the G. P. $a, ar, ar^2, \cdots$.

12. In a G. P. of 10 terms, the sum of the even terms is 30 and of the odd terms 60. Find a and r.

13. Find four numbers in G. P. such that the product of the first and last is 400 and the quotient of the middle two is 4.

14. Find three numbers in G. P. such that their sum is h, the sum of their squares k.

15. If a tree, now 4 inches in diameter, increases its diameter 5% each year, how thick will it be in 20 years?

16. A seed yields a plant from which 4 new seeds are obtained. How many seeds are available from the 10th generation of plants?

17. Radium disintegrates at the rate of about .0004 or .04% per year; that is, an amount of w grams will lose .0004 w grams during a year. How much will 1 gram lose in 1000 years? When will half of it be gone?

18. There are about 6000 stars of the 6th magnitude, that is, just visible to the naked eye. If the 7th magnitude class contains 2.5 times as many stars as the 6th, the 8th 2.5 times as many as the 7th, and so on, how many stars in the 6th to the 20th magnitudes, inclusive?

19. An Indian potentate offered to reward the inventor of the game of chess as follows: one grain of wheat for the first square on the chessboard, 2 for the second, 4 for the third, and so on, doubling each time for the 64 squares. What would be the cash value of this reward, with wheat at $1.00 a bushel, allowing a million grains to the bushel?

20. A right triangle has a hypotenuse 2 ft., angle 30°. From the vertex of the right angle a $\perp$ is dropped on the hypotenuse, forming a new right triangle which is treated similarly, and so on indefinitely. Find the sum of all the $\perp$s so obtainable.

21. The altitude of an equilateral triangle is a. A circle is inscribed in it, and in this circle a new equilateral triangle. The operation is repeated on the new triangle, and so on indefinitely. Find the sum of the altitudes and of the perimeters of all triangles so obtainable.

22. Find the sum of the perimeters and of the areas of all the circles in exercise 19.

133. Annuities. — *An annuity is a sum of money payable yearly, or at other stated periods.*

Let A be the amount of each payment, r the yearly rate of interest, n the number of payments to be made.

Amount of an Annuity. Assuming the first payment now due, and that each payment is put at interest, compounded annually, what is the total amount accrued when the last payment has been made?

The first payment is at interest $n - 1$ years, its amount, by (**47**), $A\,(1 + r)^{n-1}$; the second $n - 2$ years, its amount $A\,(1+r)^{n-2}$; and so on, to the payment next before the last, which is at interest one year, its amount $A\,(1 + r)$; the last payment amounts to A. The total amount S is therefore

$$S = A + A\,(1 + r) + A\,(1 + r)^2 + \cdots + A(1 + r)^{n-1}, \text{ or}$$

$$(1) \qquad S = A\,\frac{(1 + r)^n - 1}{1 + r - 1} = A\,\frac{(1 + r)^n - 1}{r}.$$

Present Worth of an Annuity. How much cash in hand, placed at interest compounded annually, will amount to the sum S just obtained when the last payment is made, that is, in $n - 1$ years?

Let Q be the sum required, called the *present worth* of the annuity.

In $n - 1$ years, at interest rate r compounded annually, this will amount to $Q\,(1 + r)^{n-1}$. Putting this equal to the value of S we have

$$Q\,(1 + r)^{n-1} = A\,\frac{(1 + r)^n - 1}{r}.$$

(2) Therefore $$Q = A\,\frac{(1 + r)^n - 1}{r(1 + r)^{n-1}}.$$

Present Worth of a Deferred Annuity. Let the annuity consist of n yearly payments of A dollars each, the first payment being due in m years. This is called a *deferred annuity.* Its last payment is due in $m + n - 1$ years from date.

If the successive payments were placed at interest when received, the total amount in the bank would be, as before,

$$S = A\,\frac{(1 + r)^n - 1}{r}.$$

Let Q be the sum of money required to be placed at interest *now,* such that it will amount to the sum S at the expiration of the annuity, that is, in $m + n - 1$ years.

The amount of Q dollars for $m + n - 1$ years, at interest rate r compounded annually is $Q\,(1 + r)^{m+n-1}$.

Putting this equal to the value of S and solving for Q, we have

(3) $$Q = A\,\frac{(1 + r)^n - 1}{r\,(1 + r)^{m+n-1}}.$$

Present Worth of a Perpetual Annuity. If an annuity, whose first payment is now due, is to run in perpetuity, its present worth may be found by letting n in (2) become infinite. To determine the limiting value of Q in this case, divide both sides of the fraction by $(1 + r)^{n-1}$, so that (2) becomes

$$Q = \frac{A}{r}\left[(1 + r) - \frac{1}{(1 + r)^{n-1}}\right].$$

Now as n becomes infinite, $\frac{1}{(1 + r)^{n-1}}$ becomes zero, hence

$$Q = A\frac{1+r}{r}. \tag{4}$$

This sum, if placed at compound interest, will be just sufficient to pay an annuity of A dollars indefinitely, the first payment being now due.

134. Exercises.

1. Find the present worth of an annuity of $100, there being 20 annual payments of which the first is now due. Interest 5%.

2. An annuity of $100 is to begin in 10 years from date and to run 10 years. Find its present worth if money brings 5% compound interest.

3. Find the amount and present worth of an annuity of $75, there being 15 payments of which the first is due in 5 years. Interest rate 4%.

4. Find the amount and present worth of an annuity of $250, there being 10 payments of which the first is due in 20 years. Interest 5%.

5. Find the present worth of a perpetual annuity of $250; interest rate 4%.

6. Find the present worth of a perpetual annuity of $50; interest rate 5%.

7. If a man aged 45 years has an expectation of life of 24 years, how much should he pay now to an insurance company for an annuity of $100 to begin in 10 years and continue during his life? Reckon interest at 4%; no account to be taken of profits.

8. Compare a twenty-payment life insurance policy for $1000, annual premium $48, with the amount produced if each premium had been placed at 4% compound interest instead of having been paid to the insurance company.

CHAPTER VIII

INFINITE SERIES

135. Limit of a Variable Quantity. — *When a variable quantity changes in such a way that it approaches a fixed numerical value, so that the difference between the variable and the fixed quantity becomes and remains less than any assignable magnitude, however small, then the fixed quantity is called the limit of the variable.*

For example, as x varies the variable quantity $1 + x$ can be made to differ from 1 by less than any small quantity e, by simply taking $|x| < e$, and the nearer x is to 0, the nearer will $1 + x$ be to 1. Hence, *as x approaches* 0, *the limit of* $1 + x$ *is* 1. As an equation this is expressed by

$$\lim_{x \doteq 0} (1 + x) = 1. \quad (\doteq \text{ is read ``}\textit{approaches.}\text{''})$$

Exercise. Show that:

$$(a)\ \lim_{x \doteq 0} \frac{1}{1 + x} = 1; \quad (b)\ \lim_{x \doteq 1} \left(1 + \frac{1}{x}\right) = 2; \quad (c)\ \lim_{x \doteq 0} \log (1 + x) = 0;$$

$$(d)\ \lim_{n \doteq 10} \left(1 - \frac{10}{n}\right) = 0; \quad (e)\ \lim_{x \doteq 0} e^x = 1; \quad (f)\ \lim_{n \doteq \infty} \left(1 + \frac{1}{n}\right)^{\frac{1}{n}} = 1.$$

136. Infinite Series. — *A sequence or succession of terms, u_1, u_2, u_3, $\cdots$, u_n, $\cdots$, unlimited in number, is called an infinite series.*

The sum of the first n terms of a sequence we denote by S_n. Then

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n.$$

As n increases and we form the sum of more and more terms of the sequence, one of three alternatives is open to S_n, namely:

(*a*) S_n approaches a fixed limit S, which is then called the sum of the infinite series, and the series is said to *converge.*

(*b*) S_n increases or decreases without limit; the infinite series then has no sum and is said to *diverge.*

(*c*) S_n oscillates; the infinite series has no limiting value and is again said to *diverge.*

Examples.

(*a*)
$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}+\cdots.$$

$$S_n=\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}=\frac{1}{2}\,\frac{(\frac{1}{2})^n-1}{\frac{1}{2}-1}=1-\left(\frac{1}{2}\right)^n. \quad \textbf{(197)}$$

$\therefore$ $\lim\limits_{n \doteq \infty} S_n = 1 = S$. The series converges to the value 1,

or,
$$\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}+\cdots=1. \quad [(\textbf{197}),\ \text{figure.}]$$

(*b*)
$$1+2+3+\cdots+n+\cdots.$$

$S_n = 1 + 2 + 3 + \cdots + n$; then obviously S_n increases without limit as more and more terms are added. Hence the given series has no sum, and diverges.

(*c*)
$$1-1+1-1+\cdots.$$

Here $S_1 = 1$; $S_2 = 1 - 1 = 0$; $S_3 = 1 - 1 + 1 = 1$; $S_4 = 0$; and so on indefinitely. S_n oscillates from 0 to 1 as n varies, the series is oscillatory and has no sum. We say that it diverges.

137. The Geometric Series. — *To show that an infinite series converges, it must be shown that S_n, the sum of its first n terms, approaches a definite limit as n increases indefinitely. When such limit does not exist, the series is divergent.*

The direct method of determining whether a given series converges or diverges is to form the sum of its first n terms S_n, and let n increase indefinitely. This method is applicable only in the few cases where a formula for S_n is available. The standard case is that of the infinite geometric progression,

$$a+ar+ar^2+\cdots+ar^{n-1}+\cdots.$$

Here
$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{1-r^n}{1-r}.$$

When r is numerically less than 1, i. e., $|r| < 1$, then r^n approaches 0 as n increases and

$$\lim_{n \doteq \infty} S_n = a\frac{1}{1-r} = S.$$

When $r = 1$,

$$S_n = a + a + \cdots + a = na.$$

Hence S_n increases without limit when n increases. When $|r| > 1$, r^n increases indefinitely with n; hence S_n does the same. Therefore, *the geometric series, $a + ar + ar^2 + \cdots$, converges when $|r| < 1$, and diverges when $|r| \geqq 1$.*

Putting $a = 1$, we see that the *simple power-series*, $1+x+x^2+\cdots$, *converges when* $|x| < 1$ *and diverges when* $|x| \geqq 1$.

We next consider indirect methods for establishing the convergence or divergence of a given infinite series.

138. Theorem 1. *When an infinite series converges, its nth term approaches zero as a limit when n increases.*

Proof. Let the convergent series be $u_1+u_2+u_3+\cdots+u_n+\cdots$. Then $S_n = u_1 + u_2 + \cdots + u_n$ and $S_{n-1} = u_1 + u_2 + \cdots + u_{n-1}$.

Hence $$u_n = S_n - S_{n-1}.$$

By taking n large enough, both S_n and S_{n-1} can be made to differ from the sum of the series and hence from each other by as little as we please; hence their difference, u_n, can be made to differ from zero by less than any assignable small quantity.

$$\therefore \quad \lim_{n \doteq \infty} u_n = 0.$$

This is a necessary condition for the convergence of any series.

Test for Divergence. — From *Theorem* 1 we infer that *an infinite series diverges whenever* $\lim\limits_{n \doteq \infty} u_n \neq 0$.

Alternating Series. — A series whose terms are alternately + and − is called an *alternating series.*

139. Theorem 2. *An alternating series converges provided that* (*a*) *each term is numerically less than the preceding, and* (*b*) *the limit of the nth term is zero as n increases indefinitely.*

Proof. Let the series be

$$u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + \cdots.$$

Write this in the two forms,

$$(u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \cdots;$$
$$u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots.$$

Each set of parentheses incloses a positive quantity according to condition (*a*) of the theorem; hence assuming that $u_1, u_2, u_3, \cdots$ are themselves positive quantities, the first form shows that the sum of the series is positive, i.e., > 0, and the second that the sum is less than the first term u_1. Also, since $\lim\limits_{n \doteq \infty} u_n = 0$, the sum cannot oscillate. Hence the series converges to a value between 0 and its first term.

Example. The alternating series,

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$

converges to a value between 0 and 1.

140. Absolute Convergence. — *A series is said to converge absolutely when it remains convergent if all its terms are taken positively.*

Thus if $u_1, u_2, u_3, \cdots$ be in part negative and in part positive, the series

$$u_1 + u_2 + u_3 + \cdots$$

converges absolutely provided that the series

$$|u_1| + |u_2| + |u_3| + \cdots$$

converges.

Exercise. Show that the series

$$1 + x + x^2 + \cdots \quad \text{and} \quad a + ax + ax^2 + \cdots$$

both converge absolutely when $|x| < 1$.

141. The Comparison Test.

Let $$u_1 + u_2 + u_3 + \cdots$$

be a series known to converge absolutely or to diverge.

Let $$v_1 + v_2 + v_3 + \cdots$$

be a series to be tested for convergence or divergence. Then,

(a) *If the u-series converges absolutely and, for all values of n, v_n is numerically less than u_n, the v-series also converges absolutely;*

(b) *If the u-series diverges and v_n is numerically greater than u_n, and if all the terms of the v-series have the same sign, the v-series also diverges.*

Proof.

Let $$U_n = |u_1| + |u_2| + |u_3| + \cdots + |u_n|$$
and $$V_n = |v_1| + |v_2| + |v_3| + \cdots + |v_n|.$$

Then by condition (a), U_n approaches a limit, say U, as $n \doteq \infty$, and also, $V_n < U_n$. Hence, since V_n must increase steadily with n, but is always less than U_n, it must approach a limit V, less than U. Hence the v series converges.

Under condition (b), U_n increases without limit, and also, $V_n > U_n$. Hence V_n also increases without limit and the v-series diverges, since its terms are all of like sign.

142. Standard Test Series. (For use in Comparison Test.)

$$\left.\begin{array}{ll}(1) & a+ax+ax^2+\cdots+ax^n\cdots, \\ (2) & 1+x+x^2+\cdots+x^n+\cdots,\end{array}\right\}\begin{array}{l}\text{Conv. when } |x|<1; \\ \text{Div. when } |x| \geqq 1.\end{array}$$

$$(3)\quad 1+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}+\cdots, \qquad \text{Convergent.}$$

$$(4)\quad 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots, \qquad \text{Divergent.}$$

$$(5)\quad \frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}+\cdots, \left\{\begin{array}{l}\text{Conv. when } p>1; \\ \text{Div. when } p \leqq 1.\end{array}\right.$$

The first three of these series are geometric progressions and have already been considered.

Series (4) can be shown to diverge by grouping its terms thus:

$$1+\tfrac{1}{2}+(\tfrac{1}{3}+\tfrac{1}{4})+(\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8})+(\tfrac{1}{9}+\tfrac{1}{10}+\cdots+\tfrac{1}{16})+\cdots.$$

We can form in this way an infinite number of parentheses, each of which is $> \frac{1}{2}$. Hence the sum is infinite.

Series (5) is, term for term, greater than or equal to (4), when $p \leqq 1$; hence for these values of p the series diverges, by condition (b) above. When $p > 1$, the series is shown to converge by grouping its terms as follows:

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\cdots+\frac{1}{7^p}\right)+\left(\frac{1}{8^p}+\cdots+\frac{1}{15^p}\right)+\cdots.$$

Considering each group of terms as a single quantity, we see that this series is less, term for term, than the series

$$1+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\cdots,$$

or

$$1+\frac{1}{2^{p-1}}+\frac{1}{4^{p-1}}+\frac{1}{8^{p-1}}+\cdots.$$

But this is a G. P. with ratio $\frac{1}{2^{p-1}}$, and hence converges. Therefore the given series converges.

Examples.

1. The series $1+\frac{1}{2^2}+\frac{1}{3^3}+\cdots+\frac{1}{n^n}+\cdots$ converges; for it is less, term for term, than (3).

2. The series $1+\frac{1}{\log_{10}2}+\frac{1}{\log_{10}3}+\cdots+\frac{1}{\log_{10}n}+\cdots$ diverges; for it is greater, term for term, than (4).

143. The Ratio Test. — *The series* $u_1+u_2+u_3+\cdots+u_n+\cdots$ *converges absolutely if, beginning at some point in the series, the ratio* $u_n \div u_{n-1}$ *becomes and remains numerically less than a fixed positive number which is itself less than 1.*

Proof. Assume that

$$\left|\frac{u_n}{u_{n-1}}\right| < r < 1 \text{ for all values of } n > N,$$

N being a fixed positive integer.

Then $$|u_n| < r\,|u_{n-1}| \text{ when } n > N.$$

Hence putting $n = N+1, N+2, \cdots$, we have

$$|u_{N+1}| < r\,|u_N|;$$
$$|u_{N+2}| < r\,|u_{N+1}| < r^2\,|u_N|;$$
$$|u_{N+3}| < r\,|u_{N+2}| < r^3\,|u_N|;$$

Adding, we have

$$|u_{N+1}| + |u_{N+2}| + |u_{N+3}| + \cdots < |u_N|\,(r + r^2 + r^3 + \cdots).$$

Writing the given series in two parts,

$$(u_1 + u_2 + \cdots + u_N) + (u_{N+1} + u_{N+2} + u_{N+3} + \cdots),$$

we see that the first part, formed of N terms where N is a fixed finite integer, must have a finite sum. The second part cannot exceed the left member of the last inequality above, hence is less than the right member of that inequality. But the series $r + r^2 + r^3 + \cdots$ converges and has a finite sum, since it is a G.P. with ratio $r < 1$. Hence the sum in the second pair of parentheses has a finite limit, and the given series converges.

Similarly it can be shown that the series diverges when the test-ratio $u_n \div u_{n-1}$ becomes and remains greater than 1, or even when it approaches 1 from the upper side.

When the test-ratio $u_n \div u_{n-1}$ is at first less than 1 but approaches 1 as n increases, this method gives no information about the series.

Examples.

1. $$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

$$\left|\frac{u_n}{u_{n-1}}\right| = \left|-\frac{n-1}{n}x\right| = \frac{n-1}{n}|x|.$$

If $|x|$ is taken less than or equal to r, where r is any number between 0 and 1, the above test-ratio will be less than r for all values of n. Hence x may have any value between -1 and $+1$, and the series will converge. We may add also the value $x = 1$, as can be verified easily by putting $x = 1$ in the given series. Therefore the series converges for the values $-1 < x \leqq 1$.

Note. The above series determines the *natural* logarithms of numbers, that is, logarithms to a certain base e, where $e = 2.7182818+$. We have, namely, as is shown in works on Calculus,

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots; \quad -1 < x \leqq 1.$$

It is also shown that

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots; \quad -1 \leqq x < 1.$$

Owing to the restricted range of values of x in these series, they can be used to calculate logarithms of numbers between 0 and 2 only, including 2, but excluding 0.

2. $\log_2 x + 2\log_2^2 x + 3\log_2^3 x + \cdots + n\log_2^n x + \cdots.$

$$\left|\frac{u_n}{u_{n-1}}\right| = \left|\frac{n}{n-1}\log_2 x\right| = \frac{n}{n-1}\left|\log_2 x\right|.$$

As $n \doteq \infty$, $\frac{n}{n-1} \doteq 1$, and if we choose x between $\frac{1}{2}$ and 2, so that $|\log_2 x| < 1$, we can take n so large that, for this value and for all larger values, the test-ratio will become and remain less than r, where r is less than 1. Hence the series converges for any value of x between $\frac{1}{2}$ and 2.

3. $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$ See (4) of **142.**

$\left|\frac{u_n}{u_{n-1}}\right| = \frac{n-1}{n}$, which approaches 1 from the lower side. Hence the test fails.

4. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n}{n+1} + \cdots.$

$$\left|\frac{u_n}{u_{n-1}}\right| = \frac{n}{n+1} \div \frac{n-1}{n} = \frac{n^2}{n^2-1}.$$

Here the test-ratio is greater than 1, approaching 1 from the upper side as $n \doteq \infty$. Hence the series diverges. This series may also be shown to diverge by comparison with (4) of (**142**).

144. Exercises. — Test the following series:

1. $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots.$

2. $1 + \frac{x}{2} + \frac{x^2}{3} + \cdots + \frac{x^n}{n+1} + \cdots.$

3. $1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots.$

4. $\log_{10} x + (\log_{10} x)^2 + \cdots + (\log_{10} x)^n + \cdots.$

5. $\frac{1 \cdot 2}{3 \cdot 4} + \frac{2 \cdot 3}{4 \cdot 5} + \frac{3 \cdot 4}{5 \cdot 6} + \cdots.$

6. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots.$

7.* $1!\,x + 2!\,x^2 + 3!\,x^3 + \cdots + n!\,x^n + \cdots.$

8. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$

9. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$

10. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$

11. $1 - \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}x^4 - \cdots.$

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* $n! \equiv 1 \cdot 2 \cdot 3 \cdots n.$

CHAPTER IX

COMPUTATION. APPROXIMATIONS. DIFFERENCES AND INTERPOLATION

145. Remarks on Computation. — (1) In a series of similar computations, perform similar operations together. If the same number is to be added to each of several others write it on the edge of a slip of paper and hold it over or under each number in turn.

(2) When a result is wanted to, say, three decimals, computations should be carried to four places so as to avoid accumulation of errors which would vitiate the third place.

(3) As a general rule, 4-, 5-, 6-, and 7-place logarithm tables will yield respectively not more than 4, 5, 6, or 7 significant figures of a number.

(4) Results should be stated with an accuracy commensurate with that of the data. Thus, if a line be measured 10 times to 0.01 ft., the mean of the 10 measures should be given to 0.001 ft. More than three places in the mean would be a useless refinement. Do not state an angle to seconds when it results from computations which render even the minute uncertain.

146. Useful Approximations. — Let the student verify that, when x, y, u, v are small decimals, we have *approximately:*

1. $(1+x)(1+y) = 1+x+y.$

2. $(1+x)(1-y) = 1+x-y.$

3. $(1-x)(1-y) = 1-x-y.$

4. $\dfrac{1}{1+x} = 1-x.$

5. $\dfrac{1}{1-x} = 1+x.$

6. $\dfrac{1+x}{1+y} = 1+x-y.$

7. $\dfrac{(1+x)(1+y)}{(1+u)(1+v)} = 1+x+y-u-v.$

8. $(1+x)^n = 1+nx.$

As special cases of (8) we have

9. $\sqrt{1+x} = 1+\frac{1}{2}x.$

10. $\sqrt{1-x} = 1-\frac{1}{2}x.$

11. $\dfrac{1}{\sqrt{1+x}} = 1-\frac{1}{2}x.$

12. $\dfrac{1}{\sqrt{1-x}} = 1+\frac{1}{2}x.$

13. $(1+x)^2 = 1+2x.$

14. $(1-x)^2 = 1-2x.$

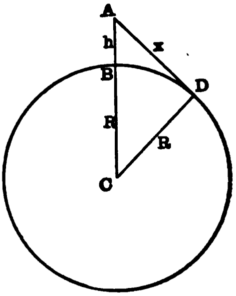
Examples.

1. $.987 \times .993 = (1 - .013)(1 - .007) = 1 - .013 - .007 = .980.$ The error is $.013 \times .007 = .000091.$

2. $\frac{1}{.987} = \frac{1}{1 - .013} = 1 + .013 = 1.013.$

3. $\sqrt{.987} = (1 - .013)^{\frac{1}{2}} = 1 - \frac{1}{2}(.013)$

$= .9935$, correct to four places.



4. Find the range of vision from a point h ft. above the surface of the earth.

Let A be the station of observation (figure), $AB = h$ ft., $BC = DC = R = 3960$ miles. Then

$$R = 3960 \times 5280 \text{ ft.}$$

$$x = \sqrt{(R+h)^2 - R^2} = \sqrt{2Rh + h^2} = \sqrt{2Rh}\sqrt{1 + \frac{h}{2R}}.$$

For moderate elevations, $\frac{h}{2R}$ is small and the second radical = 1 approximately.

Hence $$x = \sqrt{2Rh} \text{ approximately.}$$

The error in this value of x is $\frac{h}{4R}x$ approximately.

Exercises. Calculate the approximate values of,

1. $\frac{.965}{.982}$; **2.** $\frac{.85 \times 1.12}{1.15 \times .92}$; **3.** $\sqrt{1.20}$;

4. $\frac{1}{1.125}$; **5.** $\frac{1}{\sqrt{.975}}$; **6.** $(1.15)^2$.

7. Prove the last statement of example 4.

8. How far can an observer see from a mountain one mile high?

9. What is the distance to the horizon as seen by an observer on the seashore with his eye 6 ft. above the water level? (Three-mile limit.)

10. If the range of a gun on a warship is 10 miles, how high should the lookout be stationed to detect objects coming within range?

11. What is the error in each of the approximations of Exercises 1 to 14 of page 123 when $x, y, u, v = 0.1$? When $x, y, u, v = 0.01$?

147. Computation of Natural Logarithms. — Natural logarithms (see note under Example 1, p. 124) are computed by use of a modified form of the equation,

$$\log_e(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \text{ if } -1 < x \leqq 1.$$

Replace x by $-x$:

$$\log_e(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots, \text{ if } -1 \leqq x < 1.$$

Hence $\log_e(1+x) - \log_e(1-x) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right],$ provided $-1 < x < 1.$

But $\log_e(1+x) - \log_e(1-x) = \log_e \frac{1+x}{1-x}.$

Let $\frac{1+x}{1-x} = \frac{n+1}{n};$ or, $x = \frac{1}{2n+1}.$

Then $\log_e(1+x) - \log_e(1-x) = \log_e(n+1) - \log_e n$ and

$$\log_e(n+1) = \log_e n + 2\left[\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \cdots\right].$$

By means of this equation $\log(n+1)$ can be calculated when $\log n$ is known. The series on the right converges rapidly and for all positive values of n. Putting successively $n = 1, 2, 3, \cdots,$ we obtain in turn $\log 2, \log 3, \log 4, \cdots.$

We will now obtain an estimate of the maximum error made in stopping at any term of the series.

Let $k = 2n + 1.$

Then the mth term of the series is

$$u_m = \frac{1}{(2m-1)k^{2m-1}},$$

and the remainder of the series will be

$$R_m = \frac{1}{(2m+1)k^{2m+1}} + \frac{1}{(2m+3)k^{2m+3}} + \frac{1}{(2m+5)k^{2m+5}} + \cdots.$$

Then R_m is certainly less, term for term, than the series

$$\frac{1}{(2m+1)k^{2m+1}}\left[1 + \frac{1}{k^2} + \frac{1}{k^4} + \cdots\right] = \frac{1}{(2m+1)k^{2m+1}} \frac{1}{1 - \frac{1}{k^2}},$$

since the series between the brackets is an infinite G. P. with ratio $\frac{1}{k^2}$. Also, since

$k = 2n + 1$ and $n \geqq 1$, $\therefore$ $k > 2$ for all values of n. Hence

$$\frac{1}{1 - \frac{1}{k^2}} < 2$$

and therefore

$$R_m < \frac{2}{(2m+1)\,k^{2m+1}} = \frac{2}{(2m+1)\,(2n+1)^{2m+1}}.$$

If we now include the factor 2 which stands before the bracket in the equation giving log $(n + 1)$, the total error is less than

$$\frac{4}{(2m+1)\,(2n+1)^{2m+1}}$$

when log $(n + 1)$ is calculated by using only the first m terms of the series.

Thus in calculating log 5, we have $n = 5$ and the error in stopping with the mth term is less than

$$\frac{4}{(2m+1)\,11^{2m+1}}.$$

Hence when $m = 1$, the error is less than $\frac{4}{3 \cdot 11^3}$; that is, if we use only the first term of the series, log 5 will come out correct to 3 decimal places inclusive. When $m = 2$, the error is less than $\frac{4}{5 \cdot 11^5}$, so that the first two terms will give log 5 correct to 5 places, and so on.

Exercises.

1. What is the error in log 7 when only one term of the series is used? When two terms are used?

2. How many terms of the series are required to give log 7 correct to 10 places?

3. How many terms of the series are required to give log 17 to 20 places?

4. Calculate a four-place table of natural logarithms of the numbers from 1 to 20 inclusive.

148. Common Logarithms. — When the natural logarithm of a number is known, its common logarithm may be found by multiplying by a certain constant factor called the *modulus of the common system of logarithms.* We shall show that this modulus, or multiplier, is

$$M = \log_{10} e = 0.4342945+.$$

Let the natural logarithm of any number be x, its common logarithm y. To express y in terms of x. We have, if n be the number,

$$\log_e n = x \quad \text{and} \quad \log_{10} n = y,$$

or,

$$n = e^x \quad \text{and} \quad n = 10^y.$$

Hence $$10^y = e^x.$$
To solve for y, take logarithms of both members to the base 10. Then
$$y = x \log_{10} e,$$
which proves our statement. To find the value of $\log_{10} e$, we need only calculate $\log_e 10$ and take the reciprocal of the result.

Exercises.

1. Calculate the modulus M to 5 places.
2. Calculate $\log_{10} 101$ to 10 places.
3. Calculate $\log_{10} 11$ to 10 places.
4. Calculate a four-place table of common logarithms of the integers from 1 to 20 inclusive.

149. Differences. — Consider a sequence of quantities $u_0, u_1, u_2, \cdots, u_n, \cdots$, and form the differences, $\Delta u_0 = u_1 - u_0$, $\Delta u_1 = u_2 - u_1, \cdots, \Delta u_{n-1} = u_n - u_{n-1}, \cdots$, called the *first differences.* Form next the differences of these differences, called the *second differences* of the original sequence, and so on. We obtain in this way the entries in the following difference table, where the successive difference columns are denoted by $\Delta_1, \Delta_2, \Delta_3, \cdots$ and the original sequence by Δ_0.

$$\begin{array}{lllll}
\Delta_0 & \Delta_1 & \Delta_2 & & \cdots \\
u_0 & & & & \\
 & u_1 - u_0 & & & \\
u_1 & & u_2 - 2\,u_1 + u_0 & & \\
 & u_2 - u_1 & & u_3 - 3\,u_2 + 3\,u_1 - u_0 & \\
u_2 & & u_3 - 2\,u_2 + u_1 & & \cdots \\
 & u_3 - u_2 & & & \\
u_3 & & & & \\
\cdot & \cdot & & & \\
\cdot & \cdot & & & \\
\cdot & \cdot & & & \\
u_{n-2} & & & & \\
 & u_{n-1} - u_{n-2} & & & \\
u_{n-1} & & u_n - 2\,u_{n-1} + u_{n-2} & & \\
 & u_n - u_{n-1} & & & \cdots \\
u_n & & & & \\
\cdot & & & & \\
\cdot & & & & \\
\cdot & & & &
\end{array}$$

We observe that the coefficients follow the binomial law. Let the student prove by induction that this law is followed in all the successive difference columns.

150. The *n*th term of the sequence, in terms of its first term and the first terms of the first *n* difference columns.

Let the first term in the kth difference column be denoted by $\Delta_k u_0$. Then we have

$$\begin{aligned} u_0 &= u_0, \\ \Delta_1 u_0 &= u_1 - u_0, \\ \Delta_2 u_0 &= u_2 - 2\,u_1 + u_0, \\ \Delta_3 u_0 &= u_3 - 3\,u_2 + 3\,u_1 - u_0, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

Solving successively for $u_0, u_1, u_2, \cdots$, we have

$$\begin{aligned} u_0 &= u_0, \\ u_1 &= u_0 + \Delta_1 u_0, \\ u_2 &= u_0 + 2\,\Delta_1 u_0 + \Delta_2 u_0, \\ u_3 &= u_0 + 3\,\Delta_1 u_0 + 3\,\Delta_2 u_0 + \Delta_3 u_0, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

Here the coefficients again follow the binomial law, and there is suggested the formula (52)

$$u_n = u_0 + {}_nC_1\Delta_1 u_0 + {}_nC_2\Delta_2 u_0 + \cdots + \Delta_n u_0. \tag{1}$$

Assuming the formula true for u_n, we can show that it holds for u_{n+1}. For apply formula (1) to the *n*th term of the first order of differences, which is $u_{n+1} - u_n$. We obtain

$$u_{n+1} - u_n = \Delta_1 u_0 + {}_nC_1\Delta_2 u_0 + {}_nC_2\Delta_3 u_0 + \cdots + \Delta_{n+1} u_0.$$

Adding equation (1) to this we get

$$\begin{aligned} u_{n+1} = u_0 &+ ({}_nC_1 + 1)\,\Delta_1 u_0 + ({}_nC_2 + {}_nC_1)\,\Delta_2 u_0 \\ &+ ({}_nC_3 + {}_nC_2)\,\Delta_3 u_0 + \cdots + \Delta_{n+1} u_0. \end{aligned}$$

But

$${}_nC_1 + 1 = {}_{n+1}C_1,\quad {}_nC_2 + {}_nC_1 = {}_{n+1}C_2,\quad {}_nC_3 + {}_nC_2 = {}_{n+1}C_3,\quad \cdots,$$

as is easily verified by substituting in the values of the binomial coefficients. Hence

$$u_{n+1} = u_0 + {}_{n+1}C_1\Delta_1 u_0 + {}_{n+1}C_2\Delta_2 u_0 + {}_{n+1}C_3\Delta_3 u_0 + \cdots + \Delta_{n+1} u_0.$$

Hence, if (1) holds for u_n, it also holds when n is replaced by $n + 1$, that is, for u_{n+1}. But we have shown that it holds for u_3; hence it holds for u_4, hence for u_5, and so on.

151. The sum of the first n terms of the sequence, in terms of its first term and the first terms of the first $n-1$ difference columns.

From the equations just preceding formula (1) we have, by addition,

$$\begin{aligned} u_0 &= u_0, \\ u_0+u_1 &= 2\,u_0+\Delta_1u_0, \\ u_0+u_1+u_2 &= 3\,u_0+3\,\Delta_1u_0+\Delta_2u_0, \\ u_0+u_1+u_2+u_3 &= 4\,u_0+6\,\Delta_1u_0+4\,\Delta_2u_0+\Delta_3u_0. \end{aligned}$$

The coefficients on the right are respectively those of the expansions of $(1+x)^1$, $(1+x)^2$, $(1+x)^3$, and $(1+x)^4$, the first term of the expansion being omitted in each case. Let s_n denote the sum of the first n terms of the sequence;

$$s_n = u_0+u_1+u_2+\cdots+u_{n-1}.$$

Then by analogy with the preceding equations we assume that

$$(2)\quad s_n = {}_nC_1u_0+{}_nC_2\Delta_1u_0+{}_nC_3\Delta_2u_0+{}_nC_4\Delta_3u_0+\cdots+\Delta_{n-1}u_0.$$

We show by induction that (2) holds for all values of n. Adding (1) of (**150**) to (2) and noting that $s_{n+1} = s_n+u_n$, we have

$$\begin{aligned} s_{n+1} &= ({}_nC_1+1)u_0+({}_nC_2+{}_nC_1)\Delta_1u_0+({}_nC_3+{}_nC_2)\Delta_2u_0+\cdots+\Delta_nu_0 \\ &= {}_{n+1}C_1u_0+{}_{n+1}C_2\Delta_1u_0+{}_{n+1}C_3\Delta_2u_0+\cdots+\Delta_nu_0. \end{aligned}$$

Therefore (2) is true when n is replaced by $n+1$. But we verified above that (2) is true when $n = 4$. Hence it is true when $n = 5$, hence when $n = 6$, and so on.

When the *rth order of differences is zero,* all following orders of difference are also zero. Hence any term of the sequence and the sum of any number of terms can be expressed in terms of the first term of the sequence and the first terms of the first $r-1$ difference columns. For then formulas (1) and (2) both stop with the term involving $\Delta_{r-1}u_0$, and we have

$$(3)\quad u_n = u_0+{}_nC_1\Delta_1u_0+{}_nC_2\Delta_2u_0+\cdots+{}_nC_{r-1}\Delta_{r-1}u_0.$$

$$(4)\quad S_n = {}_nC_1u_0+{}_nC_2\Delta_1u_0+{}_nC_3\Delta_2u_0+\cdots+{}_nC_r\Delta_{r-1}u_0.$$

Example. Find the sum of the squares of n consecutive integers beginning with 10.

$$s_n = 10^2 + 11^2 + 12^2 + \cdots + (10 + n - 1)^2.$$

Our difference table is as follows:

Δ_0	Δ_1	Δ_2	Δ_3
100			
	21		
121		2	
	23		0
144		2	
	25		0
169		2	
	27		
196			
.			
.			
.			

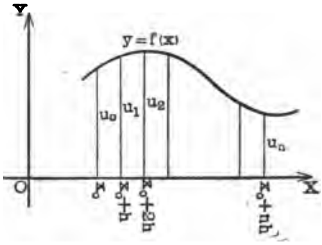
Hence $r = 3$. Then

$$\begin{aligned} S_n &= {}_nC_1u_0 + {}_nC_2\Delta_1u_0 + {}_nC_3\Delta_2u_0 \\ &= n \times 100 + \frac{n(n-1)}{1\cdot 2} \times 21 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} \times 2 \\ &= \tfrac{1}{6}(2n^3 + 57n^2 + 541n). \end{aligned}$$

Exercises.

1. Find the sum of the squares of the integers from 1 to n inclusive.
2. Find the sum of the cubes of the integers from 1 to 20 inclusive.
3. How many balls in a square pyramid whose base has n balls on a side.
4. As in exercise 3 for a triangular pyramid.
5. Find the sum of n terms of the sequence $a, a + d, a + 2d, \cdots$.
6. Find the 10th term and the $(n + 1)$th term of the sequence 50, 72, 98, 128, 162, $\cdots$. *Ans.* 392; $2n^2 + 20n + 50$.

152. Interpolation. — Suppose the terms of the sequence $u_0, u_1, u_2, \ldots$ to be the values of a function $f(x)$ for a series of equally spaced values of x. Thus:



$$\begin{aligned} u_0 &= f(x_0), \\ u_1 &= f(x_0 + h), \\ u_2 &= f(x_0 + 2h), \\ &\cdots\cdots \\ u_n &= f(x_0 + nh). \\ &\cdots\cdots \end{aligned}$$

These values are shown graphically in the figure, as ordinates of the curve $y = f(x)$. From the equally spaced ordinates given, we wish to calculate intermediate ones. This is called interpolation.

Replacing the u's in (1) of (**150**) by their values above, we have

$$f(x_0 + nh) = f(x_0) + n\Delta_1 f(x_0) + \frac{n(n-1)}{1 \cdot 2}\Delta_2 f(x_0) + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}\Delta_3 f(x_0) + \cdots. \tag{5}$$

This formula has been derived when n is a positive integer. It is also true for fractional values of n, provided the series on the right converges. We shall not stop for the proof, but merely give some simple applications. In practical cases the successive differences $\Delta_1 f(x_0)$, $\Delta_2 f(x_0)$, $\cdots$ become rapidly small, so that first differences are usually sufficient, second differences are occasionally needed, while third and higher differences are required only in theory or in the calculation of extensive tables.

For fractional values of n, formula (5) gives values of the function intermediate to those in the table. Thus when $n = 2\frac{1}{2}$, we get $f(x_0 + 2\frac{1}{2}h)$, which is the ordinate to the curve $y = f(x)$ falling midway between the ordinates $f(x_0 + 2h)$ and $f(x_0 + 3h)$.

Example 1. Given the values of log 100, log 101, $\cdots$, log 109 to five decimal places, to calculate log 100.7 and log 107.35.

Here $f(x) = \log x$; $x_0 = 100$; $h = 1$. To calculate log 100.7 we put $n = .7$. Our difference table is,

$f(x)$	$\Delta_1 f(x)$	$\Delta_2 f(x)$
log 100 = 2.00000		
	+.00432	
101 = 2.00432		−.00004
	428	
102 = 2.00860		4
	424	
103 = 2.01284		3
	421	
104 = 2.01703		5
	416	
105 = 2.02119		4
	412	
106 = 2.02531		5
	407	
107 = 2.02938		3
	404	
108 = 2.03342		3
	401	
109 = 2.03743		

Then

$$f(x_0 + nh) = \log 100.7 = \log 100 + .7 \times .00432 - \frac{.7(.7-1)}{1 \times 2} \times .00004 + \cdots$$
$$= 2 + .00302 + .00000 = 2.00302.$$

Here the second differences are so small that they can be neglected, and our result is that obtained by ordinary or linear interpolation. Graphically this amounts to replacing the curve $y = f(x)$ by its chords.

To calculate log 107.35, it is best to consider log 107 as the first term, or $f(x_0)$, and put $n = .35$. (We might take $f(x_0) = \log 100$ and put $n = 7.35$.)

We find

$$\log 107.35 = \log 107 + .35 \times .00404 - \frac{.35\,(.35-1)}{1 \times 2} \times .00003 + \cdots = 2.03079.$$

Here also second differences are negligible.

All ordinary tables are constructed so that linear interpolation is sufficient.

Example 2. Given sin 10°, sin 15°, · · · , sin 45°, to calculate sin 17° 20′.

The tabular numbers and their differences are given below:

$f(x)$	$\Delta_1 f(x)$	$\Delta_2 f(x)$	$\Delta_3 f(x)$
sin 10° = 0.1736			
	+ .0852		
15° = .2588		− .0020	
	832		− .0006
20° = .3420		26	
	806		6
25° = .4226		32	
	774		6
30° = .5000		38	
	736		6
35° = .5736		44	
	692		5
40° = .6428		49	
	643		
45° = .7071			

Here $x_0 = 10°$; $h = 5^0$; then $17° \, 20' = x_0 + \frac{22}{15} h$ and hence $n = \frac{22}{15}$. Then

$$\sin 17° \, 20' = \sin 10° + \tfrac{22}{15} \times .0852 - \frac{\frac{22}{15}\left(\frac{22}{15} - 1\right)}{1 \times 2} \times .0020$$

$$- \frac{\frac{22}{15}\left(\frac{22}{15} - 1\right)\left(\frac{22}{15} - 2\right)}{1 \times 2 \times 3} \times .0006 + \cdots = .2979.$$

Here the amount contributed by the second difference is .0003, so that linear interpolation would have been inaccurate.

153. Exercises.

1. From the table of example 1 calculate log 104.6.

2. From the table of example 2 calculate sin 12° 30′, sin 27° 30′, and sin 36° 15′.

3.

n	$\dfrac{0.6745}{\sqrt{n(n-1)}}$
10	0.0711
15	465
20	346
25	275
30	229
35	196
40	171
45	152
50	136

Calculate the tabular number when $n = 22$; when $n = 33.6$.

4.

Altitude	Refraction.
10°	5′ 13″.1
12°	4′ 22″.5
14°	3′ 45″.2
16°	3′ 16″.6
18°	2′ 54″.0
20°	2′ 35″.7
22°	2′ 20″.5
24°	2′ 7″.6
26°	1′ 56″.6

Calculate the refraction for altitudes 14° 40′ and 21° 25′.

5.

Greenwich mean time.	Moon's right ascension.	Moon's declination.
h	h m s	
0	5 14 32.14	18° 47′ 37″.7
2	5 19 49.41	18° 49′ 15″.9
4	5 25 6.62	18° 50′ 20″.6
6	5 30 23.69	18° 50′ 51″.7
8	5 35 40.59	18° 50′ 49″.4
10	5 40 57.26	18° 50′ 13″.7

Calculate the moon's right ascension and declination at $0^h\ 35^m\ 20^s$ Greenwich mean time.

6. From a four-place table take log 310, log 320, · · · , log 400. Hence calculate log 317.5.

154. Differences as a Check on Computed Values. — When a number of values of a function are calculated for equal intervals of the argument, the differences should, ordinarily, vary in a regular manner. An irregularity in one of the difference columns indicates an error in the tabular values, and often enables the computer to determine the amount of the error and so correct it.

Example.

	Δ_1	Δ_2
log 70 = 1.8451		
	+.0300	
75 = 1.8751		−.0021
	279	
80 = 1.9030		25
	254	
85 = 1.9284		4
	258	
90 = 1.9542		23
	235	
95 = 1.9777		12
	223	
100 = 2.0000		11
	212	
105 = 2.0212		

The irregularity in Δ_2 causes us to examine Δ_1; here the differences .0254 and .0258 are probably incorrect, which throws suspicion on the tabular number standing between them, namely, 1.9284. This number should evidently be larger, and by trial we find that 1.9294 is probably the correct value.

Exercises. Correct the following tables:

1. tan 15° = .268
16° = .287
17° = .306
18° = .325
19° = .344
20° = .369
21° = .384
22° = .404
23° = .425
24° = .445

2.

n	$\frac{1}{n^2}$
2.0	.250
2.2	.207
2.4	.174
2.6	.158
2.8	.127
3.0	.111
3.2	.098
3.4	.087
3.6	.077

3.

Altitude.	Refraction.
10°	5′ 13″
11°	4′ 46″
12°	4′ 22″
13°	4′ 2″
14°	3′ 45″
15°	3′ 34″
16°	3′ 16″
17°	3′ 4″
18°	2′ 54″
19°	2′ 35″

CHAPTER X

UNDETERMINED COEFFICIENTS. PARTIAL FRACTIONS

155. Theorem on Power Series. — This furnishes a useful method for expanding certain expressions in series.

If the equation

$$(1) \qquad a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots = 0$$

is true for all values of x from $x = 0$ to $x = x_0$ inclusive, where $x_0 \neq 0$, then all the coefficients are zero, that is,

$$a_0 = 0,\ a_1 = 0,\ a_2 = 0,\ \cdots,\ a_n = 0,\ \cdots.$$

Proof. Since (1) is true when $x = 0$ we have, putting 0 for x, $a_0 = 0$.

Then (1) reduces to

$$a_1x + a_2x^2 + \cdots + a_nx^n + \cdots = 0,$$

or

$$(2) \qquad x(a_1 + a_2x + \cdots + a_nx^{n-1} + \cdots) = 0.$$

This must be true for all values of x from 0 to x_0. Choose for x a value ϵ *between* 0 and x_0. Then

$$\epsilon(a_1 + a_2\epsilon + \cdots + a_n\epsilon^{n-1} + \cdots) = 0.$$

Then, since $\epsilon \neq 0$, we must have

$$a_1 + a_2\epsilon + \cdots + a_n\epsilon^{n-1} + \cdots = 0,$$

or,

$$a_1 = -\epsilon(a_2 + a_3\epsilon + \cdots + a_n\epsilon^{n-2} + \cdots).$$

The series in the last parenthesis converges, and therefore has a finite sum S. For, putting $x = \epsilon$ in (1), and omitting the first two terms, we have left the convergent series

$$a_2\epsilon^2 + a_3\epsilon^3 + \cdots + a_n\epsilon^n + \cdots,$$

and this remains convergent after division by ϵ^2. Hence

$$a_1 = -\epsilon S$$

where S depends on ϵ, but is finite for all values of ϵ between 0 and x_0. Assume now that a_1 is not equal to 0; say $a_1 = h$. We can now take ϵ so small that ϵS shall be numerically less than h; hence a_1 cannot equal h. $\therefore\ a_1 = 0$.

Then (1) reduces to

$$a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots = 0,$$

or,

$$x^2(a_2 + a_3x + \cdots + a_nx^{n-2} + \cdots) = 0.$$

Choose for x a value ϵ (not necessarily the same as ϵ above) *between* 0 and x_0. Then

$$\epsilon^2(a_2 + a_3\epsilon + \cdots + a_n\epsilon^{n-2} + \cdots) = 0.$$

Hence, since $\epsilon \neq 0$, we have

$$a_2 + a_3\epsilon + \cdots + a_n\epsilon^{n-2} + \cdots = 0,$$

or,

$$a_2 = -\epsilon(a_3 + \cdots + a_n\epsilon^{n-2} + \cdots) = 0.$$

Here again the series in parentheses converges and has a finite sum. Hence by taking ϵ sufficiently small we can show that a_2 cannot equal any number h, however small. $\therefore\ a_2 = 0$.

Similarly we show that each coefficient must be zero.

156. Theorem of Undetermined Coefficients. — If two power series in x are equal to each other for all values of x from $x = 0$ to $x = x_0$ inclusive, then the coefficients of like powers of x in the two series must be equal.

Hypothesis:

$$(1)\quad a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n + \cdots \text{ when } 0 \leqq x \leqq x_0.$$

Conclusion:

$$a_0 = b_0,\ a_1 = b_1,\ a_2 = b_2,\ \cdots,\ a_n = b_n,\ \cdots.$$

Proof. From (1), by transposition, we have

$$a_0 - b_0 + (a_1 - b_1)x + (a_2 - b_2)x^2 + \cdots + (a_n - b_n)x^n + \cdots = 0.$$

Hence by the preceding theorem,

$$a_0 - b_0 = 0,\ a_1 - b_1 = 0,\ a_2 - b_2 = 0,\ \cdots,\ a_n - b_n = 0.$$

Hence the conclusion stated above.

Corollary. The theorem remains true when either or both of the infinite series reduce to polynomials. We consider a polynomial of m terms as an infinite series in which all coefficients after the mth are zero.

Example 1. Develop $\dfrac{1 - x^2}{1 + x - x^2}$ into a power series.

Assume $$\frac{1 - x^2}{1 + x - x^2} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

Clearing, and writing the coefficients of like powers of x in vertical columns, we have

$$\begin{array}{r|r|r|l} 1 - x^2 = a_0 + a_1 & x + a_2 & x^2 + a_3 & x^3 + \cdots \\ a_0 & + a_1 & + a_2 & \\ & - a_0 & - a_1 & \end{array}$$

Equating coefficients of like powers of x, we have

$$\begin{aligned} a_0 &= 1, & \text{or} \quad a_0 &= 1, \\ a_1 + a_0 &= 0, & a_1 &= -1, \\ a_2 + a_1 - a_0 &= -1, & a_2 &= 1, \\ a_3 + a_2 - a_1 &= 0, & a_3 &= -2. \\ \cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \end{aligned}$$

Hence $$\frac{1 - x^2}{1 + x - x^2} = 1 - x + x^2 - 2x^3 + \cdots.$$

Example 2. Develop $\dfrac{1 + 2x}{6x - 5x^2 + x^3}$ into a power series.

If we put
$$\frac{1 + 2x}{6x - 5x^2 + x^3} = a_0 + a_1 x + a_2 x^2 + \cdots,$$

clear of fractions and equate coefficients, we have to begin with $1 = 0$. This absurdity results from the fact that we have not taken a proper form for the development. By inspection we see that the quotient of $1 + 2x$ divided by $6x - 5x^2 + x^3$ should start with $\dfrac{1}{6x}$. To obtain the development we put

$$\frac{1 + 2x}{6x - 5x^2 + x^3} = \frac{1}{x} \cdot \frac{1 + 2x}{6 - 5x + x^2}.$$

Developing the last fraction as in example 1,

$$\frac{1 + 2x}{6 - 5x + x^2} = \frac{1}{6} + \frac{17}{36}x + \frac{79}{216}x^2 + \frac{293}{1296}x^3 + \cdots.$$

Hence
$$\frac{1 + 2x}{6x - 5x^2 + x^3} = \frac{1}{6x} + \frac{17}{36} + \frac{79}{216}x + \frac{293}{1296}x^2 + \cdots.$$

Exercises.

1. In example 1, find a_n in terms of a_{n-1} and a_{n-2}.

Find the first four terms of the expansions of:

2. $\dfrac{1 + x}{1 + x + x^2}$.

3. $\dfrac{1 - x}{1 - x - x^2}$

4. $\dfrac{x}{2 - x + 3x^2}$.

5. $\dfrac{2x^2 + 3x}{x^2 + 2x + 2}$

6. $\dfrac{1 + x^2}{1 + 3x + x^3}$.

7. $\dfrac{2 - 3x + x^2}{3x + 4x^2 - x^3}$.

156. Partial Fractions. — It is sometimes desirable to resolve a given rational fraction into a sum of simpler fractions, called *partial fractions.* This can be done when the denominator of the given fraction can be factored. Several cases arise, according to the nature of these factors.

For reasons which will presently appear, the methods to be explained apply only to fractions in which *the degree of the numerator is less than the degree of the denominator.* When this is not the case, divide numerator by denominator until a remainder of less degree than the denominator is obtained.

Case 1. The denominator can be factored into linear factors of the form $(ax + b)$, no two factors being equal.

Rule. The fraction can be resolved into a sum of simple fractions, of the form $\frac{A}{ax+b}$, equal in number to the factors of the given denominator. Here A is a constant.

Example. $$\frac{5x-1}{x^2-6x+5} = \frac{5x-1}{(x-1)(x-5)} = \frac{A}{x-1} + \frac{B}{x-5}.$$

Clearing: $5x - 1 = A(x-5) + B(x-1)$,

or, $5x - 1 = (A+B)x - (5A+B)$.

Since the given fraction must be equal to its partial fractions for all values of x, except $x = 1$ and $x = 5$, the last equation must be true for all such values of x; hence we equate coefficients of like powers of x (**156,** Corollary). We obtain

$$5 = A + B; \quad -1 = -(5A + B).$$

Hence $A = -1; \quad B = 6.$

$$\therefore \quad \frac{5x-1}{x^2-6x+5} = \frac{-1}{x-1} + \frac{6}{x-5}.$$

A shorter method for finding A and B is as follows: consider again the equation

$$5x - 1 = A(x-5) + B(x-1).$$

Let $x = 5; \quad 24 = 4B; \quad B = 6.$

Let $x = 1; \quad 4 = -4A; \quad A = -1.$

We can justify the use of the values $x = 1$ and $x = 5$, for which the given fraction and one of the partial fractions become infinite. For the equation

$$\frac{5x-1}{x^2-6x+5} = \frac{A}{x-1} + \frac{B}{x-5}$$

must hold except when $x = 1$ or $x = 5$.

Hence

$$5x - 1 = A(x-5) + B(x-1)$$

is true for all values of x, except perhaps $x = 1$ and $x = 5$. It is therefore true when $x = 1 + \epsilon$, however small ϵ may be; that is,

$$(1) \qquad 5\,(1 + \epsilon) - 1 = A\,(1 + \epsilon - 5) + B\,(1 + \epsilon - 1).$$

Suppose our equation is not true when $x = 1$; let the two members differ by a quantity h, so that

$$5 \times 1 - 1 = A\,(1 - 5) + B\,(1 - 1) + h,$$

or,

$$4 = -\,4\,A + h.$$

From (1) we have

$$4 + \epsilon = -\,4\,A + \epsilon A + \epsilon B.$$

From the last two equations, by subtraction, etc.,

$$h = \epsilon\,(A + B - 1).$$

Since A and B are fixed numbers, h can be made as small as we wish by taking ϵ small enough. Hence h cannot equal any number except 0.

157. Case 2. — The denominator contains a linear factor repeated r times, as $(ax + b)^r$.

Rule. Corresponding to the factor $(ax + b)^r$, take a set of partial fractions of the form

$$\frac{A_1}{(ax + b)} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_r}{(ax + b)^r}.$$

This is the most general set of fractions having constant numerators and common denominator $(ax + b)^r$.

Example.

$$\frac{3\,x^2 - x + 1}{(x + 2)\,(x - 3)^3} = \frac{A}{x + 2} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2} + \frac{D}{(x - 3)^3}.$$

Clearing:

$$3\,x^2 - x + 1 = A\,(x - 3)^3 + B\,(x + 2)\,(x - 3)^2 + C\,(x + 2)\,(x - 3) + D\,(x + 2).$$

Let $x = 3$; then $25 = 5\,D$; $D = 5$.
Let $x = -\,2$; then $15 = -\,125\,A$; $A = -\frac{3}{25}$.

Since no other factors are available to furnish other values of x for substitution, we choose any convenient values, say $x = 0$ and $x = 1$.

Put $x = 0$; $1 = -\,27\,A + 18\,B - 6\,C + 2\,D$.
Put $x = 1$; $3 = -\,8\,A + 12\,B - 6\,C + 3\,D$.

Substituting the values of A and D already found, and solving for B and C, we have

$$B = \tfrac{3}{25}; \quad C = \tfrac{12}{5}.$$

Hence

$$\frac{3\,x^2 - x + 1}{(x + 2)\,(x - 3)^3} = \frac{-\,3}{25\,(x + 2)} + \frac{3}{25\,(x - 3)} + \frac{12}{5\,(x - 3)^2} + \frac{5}{(x - 3)^3}.$$

158. Case 3. — **The denominator contains a quadratic factor, $(ax^2 + bx + c)$, which cannot be resolved into real linear factors.**

Rule. Corresponding to a quadratic factor $(ax^2 + bx + c)$, take a partial fraction of the form

$$\frac{Ax + B}{ax^2 + bx + c}.$$

The reason for this assumption may be illustrated by a simple example.

Example. Resolve $\dfrac{2x - 1}{(x - 1)(x^2 + 4)}$ into partial fractions.

If $i = \sqrt{-1}$, the factors of $x^2 + 4$ are $x + 2i$ and $x - 2i$. Suppose now we assume

$$\frac{2x - 1}{(x - 1)(x^2 + 4)} = \frac{A}{x - 1} + \frac{B}{x + 2i} + \frac{C}{x - 2i}.$$

Combining the last two fractions into a single one, we have

$$\frac{B}{x + 2i} + \frac{C}{x - 2i} = \frac{(B + C)x + 2(C - B)i}{x^2 + 4}.$$

If now we introduce two new constants M, N in place of B, C, by the relations

$$B = M + iN; \quad C = M - iN,$$

we have

$$B + C = 2M; \quad i(C - B) = -2i^2N = 2N.$$

Hence in place of the fractions

$$\frac{B}{x + 2i} + \frac{C}{x - 2i},$$

where B and C involve i, we take the single fraction

$$\frac{Mx + 4N}{x^2 + 4},$$

where M and N are real. Then, using B in place of M and C in place of $4N$, let

$$\frac{2x - 1}{(x - 1)(x^2 + 4)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 4}.$$

Clearing: $\quad 2x - 1 = A(x^2 + 4) + (Bx + C)(x - 1).$

Put $\quad x = 1;$ then $\quad 1 = 5A; \quad A = \frac{1}{5}.$

Put $\quad x = 0;$ then $\quad -1 = 4A - C; \quad C = \frac{9}{5}.$

Equate coefficients of x^2; then $\quad 0 = A + B; \quad B = -A = -\frac{1}{5}.$

Hence

$$\frac{2x - 1}{(x - 1)(x^2 + 4)} = \frac{1}{5(x - 1)} + \frac{-x + 9}{5(x^2 + 4)}.$$

159. Case 4. — The denominator contains a repeated quadratic factor, $(ax^2 + bx + c)^r$.

Rule. Corresponding to a repeated quadratic factor $(ax^2 + bx + c)^r$, take the partial fractions,

$$\frac{B_1x + C_1}{(ax^2 + bx + c)} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_rx + C_r}{(ax^2 + bx + c)^r}.$$

Example.

$$\frac{10x^3 + 7x + 4}{(x-2)(x^2+3)^2} = \frac{A}{x-2} + \frac{Bx + C}{x^2+3} + \frac{Dx + E}{(x^2+3)^2}.$$

Clearing:

$$10x^3 + 7x + 4 = A(x^2+3)^2 + (Bx + C)(x-2)(x^2+3) + (Dx + E)(x-2).$$

Put $x = 2;\quad 98 = 49A;\quad A = 2.$

Equate coefficients of x^4, x^3, x^2, and x^0:

$$\begin{aligned} 0 &= A + B, \\ 10 &= C - 2B, \\ 0 &= 6A + 3B - 2C + D, \\ 4 &= 9A - 6C - 2E. \end{aligned}$$

Hence $B = -2,\ C = 6,\ D = 6,\ E = -11.$

Therefore

$$\frac{10x^3 + 7x + 4}{(x-2)(x^2+3)^2} = \frac{2}{x-2} + \frac{-2x + 6}{x^2+3} + \frac{6x - 11}{(x^2+3)^2}.$$

160. Exercises. — Resolve into partial fractions:

1. $\dfrac{1}{3x^2 + 10x + 3}.$

2. $\dfrac{3x - 1}{x^2 + x - 6}.$

3. $\dfrac{x^2 + 6x - 8}{x^3 - 4x}.$

4. $\dfrac{1 + x^2}{x - x^3}.$

5. $\dfrac{\text{[illegible]}\,x}{x^2 - 4x + 1}.$

6. $\dfrac{x^4}{x^3 + 2x^2 - x - 2}.$

7. $\dfrac{1}{x^4 - 1}$

8. $\dfrac{x^5 + x^4 - 8}{x^3 - 4x}.$

9. $\dfrac{5x + 12}{x^3 + 4x}.$

10. $\dfrac{1}{(x^2 - 1)^2}.$

11. $\dfrac{x^3 - 1}{x^3 + 3x}.$

12. $\dfrac{x^3 + 1}{x(x-1)^3}.$

13. $\dfrac{x - 8}{x^3 - 4x^2 + 4x}.$

14. $\dfrac{1}{x^4 + x^3 + x^2 + x}.$

15. $\dfrac{1}{x^3 + 1}.$

16. $\dfrac{x^2 - 1}{x^2 - 4}.$

17. $\dfrac{x^2 - 3}{x^3 - 7x + 6}.$

18. $\dfrac{x^3 - 2x + 1}{x^4 + 2x^3 + x^2}.$

19. $\dfrac{3x^2 - 2x}{x^3 - 3x^2 + 2x}.$

20. $\dfrac{x^2 + 3x + 4}{x^3 + 2x^2 + x}.$

21. $\dfrac{x^2 + 8x + 4}{x^3 + x^2 - 4x - 4}.$

22. $\dfrac{x^3 - 2x - 1}{x^4 + 3x^2 + 2}.$

CHAPTER XI

DETERMINANTS

161. Determinants of the Second Order. — When two simultaneous linear equations

$$a_1x + b_1y = c_1,$$
$$a_2x + b_2y = c_2,$$

are solved for x and y, we find

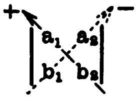
$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}; \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

To express these results it is convenient to use the notation

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = (a_1b_2 - a_2b_1),$$

where the square array between vertical bars is simply another way of writing the expression forming the right member of the equation. It is called a determinant, and in particular, a determinant of the second order, because there are two rows and two columns. The quantities a_1, b_1, a_2, b_2, are called the elements of the determinant.

The value of a determinant of the second order may be obtained by forming the products of elements which constitute the diagonals of the array and giving these products the signs indicated in the scheme below:



This process is called "expanding the determinant."

The above values of x and y may now be written in the forms,

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Exercises.

1. State a rule for writing the above values of x and y.

Solve for x and y, by aid of determinants:

2. $x - y = 1,$ $\quad 2x + y = 3.$

3. $4x - 3y = 5,$ $\quad 2x + y = 1.$

4. $8x + 5y - 6 = 0,$ $\quad 4x + y + 4 = 0.$

5. $2x + y + 1 = 0,$ $\quad 6x + 3y + 2 = 0.$

6. $2x + y + 1 = 0,$ $\quad 6x + 3y + 3 = 0.$

162. Determinants of the Third Order. — We shall now define a determinant of the third order in terms of determinants of the second order by the following equation:

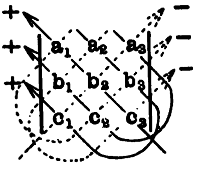
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

where the determinants on the right are to be expanded and the results multiplied by the quantity written in front of the determinants respectively.

On performing these operations and collecting terms, we have

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2.$$

This is the expanded form of a determinant of the third order, and may be written out by forming the products of the terms joined by arrows in the scheme below, each product to be given the sign indicated.



We may now verify by direct calculation that the values of x, y, z, obtained by solving the linear equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2, \\ a_3x + b_3y + c_3z &= d_3, \end{aligned}$$

are,

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Exercises.

1. Verify the last statement.

2. State a rule for solving three equations of the form just considered.

Solve the following systems of equations:

3. $\begin{aligned} x - y + z &= 1, \\ x + 2y + 3z &= 2, \\ 3x + 2y + z &= 3. \end{aligned}$

4. $\begin{aligned} 2x + 2y - z &= 2, \\ x + y - 2z &= 1, \\ x - y + z &= 4. \end{aligned}$

5. $\begin{aligned} 5x + 6y - 3z &= 4, \\ 4x - 5y + 2z &= 3, \\ 2x - 3y + z &= 1. \end{aligned}$

6. $\begin{aligned} 3x - 6y + 9z &= 2, \\ x + y + z &= 1, \\ x - 2y + 3z &= 2. \end{aligned}$

7. $\begin{aligned} x - y + z &= 2, \\ 2x + y + 3z &= 1, \\ 2x - 2y + 2z &= 4. \end{aligned}$

8. $\begin{aligned} 2x - y + 2z &= 2, \\ x - 2y + 4z &= 3, \\ 3x - 3y + 6z &= 1. \end{aligned}$

9. Show that a determinant of second or third order vanishes when the elements of a row or column are equal respectively to those of another row or column.

10. Show that a determinant changes sign when the signs of all the elements of any row or column are changed.

11. Show that, if the elements of any row or column be multiplied by a factor k, the determinant is multiplied by k.

163. Inconsistent or Non-independent Linear Equations. — Consider the equations

$$a_1x + b_1y = c_1 \quad \text{and} \quad ka_1x + kb_1y = c_2.$$

These are inconsistent if $c_2 \neq kc_1$; they are dependent if $c_2 = kc_1$, since in this case the second equation is k times the first.

In either case the determinant of the coefficients of x and y is 0. On solving by the determinant method, we find

$$x = \infty \text{ and } y = \infty, \text{ when the equations are inconsistent;}$$

$$x = \frac{0}{0} \quad \text{and} \quad y = \frac{0}{0}, \text{ when the equations are dependent.}$$

That is, the inconsistent equations have no (finite) solution, while the solution is indeterminate in case of dependent equations.

Geometrically, the equations represent two straight lines which are parallel, and distinct if $c_2 \neq c_1k$; they coincide if $c_2 = c_1k$.

Hence the infinite values of x and y above are equivalent to the statement, "Parallel lines meet at infinity." In the second case, when the lines coincide, the coördinates of any point on either line satisfy both equations. Hence there are an infinite number of solutions, and hence x and y appear above as indeterminate forms. [See exercises 5 and 6 of (**161**).]

Exercises. **1.** Consider the equations

$$a_1x + b_1y + c_1z = d_1, \quad ka_1x + kb_1y + kc_1z = d_2, \quad a_3x + b_3y + c_3z = d_3.$$

The first two are inconsistent if $d_2 \neq kd_1$, and dependent when $d_1 = kd_1$. Show that in the first case the only possible solutions of the three equations are infinite, and in the second case there is an infinite number of solutions.

2. Show that the equations

$$a_1x + b_1y = 0 \quad \text{and} \quad a_2x + b_2y = 0$$

have one solution (0, 0), or an infinite number of solutions, according as the determinant of the coefficients is different from or equal to 0. Discuss also geometrically.

3. Show that the equations

$$a_1x + b_1y + c_1z = 0, \quad a_2x + b_2y + c_2z = 0, \quad a_3x + b_3y + c_3z = 0.$$

have one solution (0, 0, 0), or an infinite number of solutions, according as the determinant of the coefficients is different from or equal to zero.

(*Hint.* Eliminate z so as to get two equations in x and y and discuss these as in exercise 2.)

4. Show that the equations

$$2x - 3y + 5z = 0, \quad x + y - z = 0, \quad 3x - 7y + 11z = 0$$

are not independent. What is the relation between them?

(*Hint.* To find the relation between the equations, find k_1 and k_2 such that k_1 times the first trinomial plus k_2 times the second shall equal the third.)

164. General Definition of a Determinant. — The array of n rows and n columns,

$$\begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ b_1 & b_2 & b_3 & \cdots & b_n \\ c_1 & c_2 & c_3 & \cdots & c_n \\ \cdot & \cdot & \cdot & & \cdot \\ l_1 & l_2 & l_3 & \cdots & l_n \end{vmatrix}$$

is called a determinant of order n. The quantities forming the array are called the elements of the determinant.

If we form all possible products of n elements, each product to contain one and only one element from each row and column, and if these products are given proper signs, as will presently be indicated, and added algebraically, the sum so obtained is defined to be the value of the determinant.

Each product of n elements so obtained is called a term of the expanded form of the determinant.

The elements $a_1, b_2, c_3, \cdots, l_n$ form the principal diagonal.

The term $a_1b_2c_3 \cdots l_n$ is called the principal term of the expansion.

165. *Every term of the expansion of the determinant can be formed from the principal term by rearranging the subscripts, leaving the letters in their natural order.*

For every term contains all the letters and all the subscripts, and each only once, since it is a product containing one and only one element from each row and each column. Hence if the letters in any term be arranged in their natural order, the subscripts will form some arrangement of the numbers 1, 2, 3, $\cdots$, n.

Conversely, every rearrangement of subscripts in the principal term, the letters being left in their natural order, yields a term of the expansion, since it contains one element and only one from each row and each column.

Therefore all the terms of the expansion can be obtained by forming all possible arrangements of subscripts in the principal term.

We shall use the symbol Δ_n to indicate our determinant of order n. Then we can write the equation

$$\Delta_n = \Sigma \pm a_1b_2c_3 \cdots l_n, \qquad (\Sigma = \text{sigma})$$

where the symbol Σ (sign for a sum) means that we are to form the algebraic sum of all terms which may be formed from the term written by forming all possible arrangements of the subscripts; the signs of the terms so formed remain to be determined.

166. Number of Terms in the Expansion of Δ_n. — The number of terms in the expansion of determinant of order n is $1 \times 2 \times 3 \times \cdots \times n$, or $n!$

Proof. We need only show that the number of possible arrangements of the subscripts 1, 2, 3, $\cdots$ n, is $n!$

Starting with the natural order, and interchanging 1 in turn with 2, 3, $\cdots$, n, we form the n arrangements

$$\begin{array}{l} 1\ 2\ 3 \cdots n, \\ 2\ 1\ 3 \cdots n, \\ 2\ 3\ 1 \cdots n, \\ \cdot\quad \cdot\quad \cdot\quad \cdot \\ 2\ 3\ 4 \cdots 1. \end{array}$$

In any one of these, keep 1 fixed in its position, and interchange 2 with 3, 4, $\cdots$, n. In this way we form $n - 1$ arrangements in which 1 occupies a given place. Treating each of the n arrangements written above similarly, we obtain altogether $n(n-1)$ arrangements. Each of these gives rise to a group of $n - 2$ arrangements, including itself, by interchanging 3 with 4, 5, $\cdots$, n. Hence we obtain $n(n-1)(n-2)$ arrangements. Proceeding similarly we find the total number of arrangements to be $n!$

167. Signs of the Terms in the Expansion of Δ_n.

Permutation. An arrangement of the numbers 1, 2, 3, $\cdots$, n is called a permutation. A permutation is even or odd according as the number of times a greater number precedes a lesser number is even or odd.

Thus, the possible permutations of 3 numbers are

123, 213, 231, 321, 312, 132;

of these the first, third, and fifth are even, the others odd.

Further, the permutations of the subscripts in the term $a_4b_2c_3d_1$ is even. For 4 precedes 2, 3, and 1, and 3 precedes 1, making a greater subscript precede a lesser one 4 times.

We now define the sign of each term of the expansion of Δ_n by the rule that the sign shall be plus when the permutation of the subscripts is even, minus when the permutation is odd.

Our determinant is now completely defined.

Exercise. Write out the expansion of

$$\Delta_4 = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

168. Properties of Determinants.

1. *A determinant is unchanged in value when its rows and columns are interchanged.*

For the expansion remains unaltered.

2. *Interchanging two adjacent rows or columns changes the sign of the determinant.*

For each term of the expansion will change sign, since two adjacent subscripts will be interchanged; hence even permutations change to odd, and vice versa.

By repeated application of this rule it follows that *if any two rows or any two columns be interchanged, the sign of the determinant changes.*

3. *If all the elements of a row or column are* 0, *the determinant* = 0.

For each term of the expansion contains a zero factor.

4. *When all the elements of a row, or column, contain a common factor, this may be taken out and written as a factor of the whole determinant.*

For each term of the expansion will contain this factor.

It follows that, *to multiply a determinant by any factor, we need only multiply the elements of any row or column by this factor.*

5. *If two rows or columns are alike, the determinant* = 0.

For by interchanging them we would have $\Delta_n = -\Delta_n$; $\therefore\ \Delta_n = 0$.

6. *If the elements of two rows or columns differ only by a common factor, the determinant* = 0.

For by taking out the common factor the two rows or columns become equal.

7. *If in the expansion of Δ_n we collect the terms which contain the several elements of any row or column, say the jth row, we have*

$$\Delta_n = \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ b_1 & b_2 & b_3 & \cdots & b_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ j_1 & j_2 & j_3 & \cdots & j_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ l_1 & l_2 & l_3 & \cdots & l_n \end{vmatrix} = j_1J_1 + j_2J_2 + \cdots j_nJ_n.$$

Here J_1 is called the cofactor of the element j_1, and similarly for $J_2, \cdots, J_n$.

8. *A determinant is unaltered in value when the elements of any row are increased by a constant multiple of the corresponding elements of another row. Similarly for columns.*

For suppose that we add to the elements of the first row k times the elements of the second. We obtain the determinant

$$\Delta_n' = \begin{vmatrix} a_1 + kb_1, & a_2 + kb_2, & \cdots, & a_n + kb_n \\ b_1 & b_2 & \cdots & b_n \\ c_1 & c_2 & \cdots & c_n \\ \cdot & \cdot & \cdot & \cdot \\ l_1 & l_2 & \cdots & l_n \end{vmatrix}$$

Let $A_1, A_2, \cdots, A_n$ be the cofactors of the elements of the first row, so that

$$\begin{aligned} \Delta_n' &= (a_1 + kb_1) A_1 + (a_2 + kb_2) A_2 + \cdots + (a_n + kb_n) A_n \\ &= (a_1A_1 + a_2A_2 + \cdots + a_nA_n) + \\ &\qquad\qquad k\,(b_1A_1 + b_2A_2 + \cdots + b_nA_n). \end{aligned}$$

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ \cdot & \cdot & \cdot & \cdot \\ l_1 & l_2 & \cdots & l_n \end{vmatrix} + k \begin{vmatrix} b_1 & b_2 & \cdots & b_n \\ b_1 & b_2 & \cdots & b_n \\ \cdot & \cdot & \cdot & \cdot \\ l_1 & l_2 & \cdots & l_n \end{vmatrix}$$

The first of these determinants is Δ_n, the second equals 0.

$$\therefore \qquad \Delta_n' = \Delta_n.$$

It follows that we can add to the elements of any row any linear combination of corresponding elements of other rows.

Example. Without expanding, show that

$$\begin{vmatrix} 102 & 104 & 106 \\ 99 & 98 & 97 \\ 1 & 2 & 3 \end{vmatrix} = 0.$$

Subtract the second row from the first. The new form is

$$\begin{vmatrix} 3 & 6 & 9 \\ 99 & 98 & 97 \\ 1 & 2 & 3 \end{vmatrix}$$

This is zero, by 6.

9. *If the cofactors of any row or column be multiplied by the elements of any other row or column, the sum of the products is zero.*

For we have

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ \cdot & \cdot & \cdot & \cdot \\ l_1 & l_2 & \cdots & l_n \end{vmatrix} = a_1A_1 + a_2A_2 + \cdots + a_nA_n.$$

Replace the a's by the elements of any other row, as the second. The result is

$$\begin{vmatrix} b_1 & b_2 & \cdots & b_n \\ b_1 & b_2 & \cdots & b_n \\ \cdot & \cdot & \cdot & \cdot \\ l_1 & l_2 & \cdots & l_n \end{vmatrix} = b_1A_1 + b_2A_2 + \cdots + b_nA_n = 0.$$

10. *If we strike out from Δ_n the jth row and kth column, the remaining determinant, of order $n - 1$, is designated by $\Delta_{j,k}$, and is called the minor of the element standing at the intersection of the row and the column struck out.*

Thus the minors of a_1, a_2, and a_3 in the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

are, respectively,

$$\Delta_{1,1} = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}, \quad \Delta_{1,2} = \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}, \quad \Delta_{1,3} = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

We shall now show that, *except as to sign, the minor of any element equals the cofactor of that element.* We shall consider a determinant of third order, although the argument will apply to determinants of any order. We have

$$\Delta_3 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1A_1 + a_2A_2 + a_3A_3,$$

where A_1, A_2, A_3 are the cofactors of a_1, a_2, a_3, respectively.

Then $$A_1 = \Delta_{1,1}.$$

For, since a_1A_1 contains all the terms of Δ_3 which involve a_1, and since the expansion of $\Delta_{1,1}$ contains all possible products of elements, one from each column and each row except the first, therefore A_1 and $\Delta_{1,1}$ must be identical. Now interchange the first two columns, so that Δ_3 becomes $-\Delta_3$. Then

$$-\Delta_3 = \begin{vmatrix} a_2 & a_1 & a_3 \\ b_2 & b_1 & b_3 \\ c_2 & c_1 & c_3 \end{vmatrix} = -a_2A_2 - a_1A_1 - a_3A_3.$$

The minor of a_2 is unchanged, namely $\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$. The expansion of this multiplied by a_2 gives all the terms of the expansion of $-\Delta_3$ containing a_2. But these are also contained in $-a_2A_2$. Hence $\Delta_{1,2} = -A_2$, or $A_{1,2} = -\Delta_{1,2}$.

In the same way, by moving the third column into first place by two successive interchanges, which does not alter the sign of the determinant, we find $\Delta_{1,3} = A_3$.

Let $A_{j,k}$ denote the cofactor of the element standing at the intersection of the jth row and kth column of Δ_n; we can bring this element to the intersection of the first row and column by $j - 1 + k - 1$ successive interchanges of rows and columns. Hence Δ_n will become $(-1)^{j+k-2} \cdot \Delta_n$ or $(-1)^{j+k}\Delta_n$, since $(-1)^{-2} = 1$; hence by reasoning as above we find

$$A_{j,k} = (-1)^{j+k}\Delta_{j,k}.$$

11. *We can now expand* Δ_n *according to the elements of its first row in the form*

$$\Delta_n = a_1\Delta_{1,1} - a_2\Delta_{1,2} + a_3\Delta_{1,3} - a_4\Delta_{1,4} + \cdots + (-1)^{n-1}\Delta_{1,n}.$$

To expand Δ_n according to the elements of any other row, we can move this row into first place and then apply the last formula.

By this rule we can express a determinant of order n in terms of determinants of order $n - 1$. Hence by repeated application of the rule we can write out the complete expansion.

By a similar process the determinant can be expanded according to the elements of any column.

169. Solution of Systems of Linear Equations. — We shall illustrate the method of solving a system of n linear equations involving n unknowns by considering three such equations with three unknowns.

Solve for x, y, and z the system of equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2, \\ a_3x + b_3y + c_3z &= d_3. \end{aligned}$$

Let the determinant of the coefficients be denoted by Δ, so that

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Let the cofactors of a_1, a_2, a_3 be A_1, A_2, A_3 respectively.

Multiply the given equations in order by A_1, A_2, A_3, and add the results. We obtain

$$(a_1A_1 + a_2A_2 + a_3A_3)\,x + (b_1A_1 + b_2A_2 + b_3A_3)\,y + \\ (c_1A_1 + c_2A_2 + c_3A_3)\,z = d_1A_1 + d_2A_2 + d_3A_3.$$

From (7) and (9) of (**168**) we see that the coefficient of x is Δ, and of y and z zero. Hence we get

$$x = \frac{d_1A_1 + d_2A_2 + d_3A_3}{\Delta} = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Similarly, by multiplying by the cofactors of b_1, b_2, b_3 and adding we get y, and by multiplying by the cofactors of c_1, c_2, c_3 and adding we get z. The results are as given in (**162**).

In precisely the same way we can solve n linear equations in n unknowns.

The exceptional cases which arise when Δ, the determinant of the coefficients, is zero, have been considered in (**163**) for the case of two and three equations. A similar discussion applies to the case of n equations.

When the equations are homogeneous (i.e., $d_1 = 0$, $d_2 = 0$, $d_3 = 0 \cdots$), and $\Delta \neq 0$, the only solution is $x = 0$, $y = 0$, $z = 0$, $\cdots$; when $\Delta = 0$, there exists an infinite number of solutions.

170. Exercises. — Evaluate the following determinants:

1. $\begin{vmatrix} a & 1 & 3 \\ a+1 & 2 & 2 \\ a+2 & 3 & 1 \end{vmatrix}$

2. $\begin{vmatrix} a & h & g \\ h & g & f \\ g & f & c \end{vmatrix}$

3. $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$

4. $\begin{vmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 2 & 10 \\ 0 & 3 & 2 & 4 \\ 6 & 2 & 0 & 3 \end{vmatrix}$

5. $\begin{vmatrix} 1 & 1 & 1 & 5 \\ 0 & 0 & 1 & 2 \\ 3 & 4 & -3 & 2 \\ 1 & -1 & 4 & 5 \end{vmatrix}$

6. $\begin{vmatrix} 6 & 2 & 8 & 2 \\ 2 & 2 & 8 & 5 \\ 1 & 6 & 4 & 2 \\ 3 & 2 & 5 & 3 \end{vmatrix}$

7. $\begin{vmatrix} 3 & 1 & 1 & 2 \\ 1 & 5 & 0 & 3 \\ 2 & -2 & 1 & 6 \\ 0 & 4 & -5 & 3 \end{vmatrix}$

8. $\begin{vmatrix} 2 & 5 & 4 \\ 4 & 1 & 6 \\ 1 & -3 & 7 \end{vmatrix}$

9. $\begin{vmatrix} 0 & 4 & 4 & 4 \\ -1 & -9 & -1 & 9 \\ -1 & 7 & 1 & -1 \\ 9 & 16 & 27 & 23 \end{vmatrix}$

10. $\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix}$

11. $\begin{vmatrix} a_1 & 0 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$

12. Show, without expanding, that

$$\begin{vmatrix} 6 & 1 & -7 \\ 5 & -10 & 5 \\ 4 & 3 & -7 \end{vmatrix} = 0.$$

13. Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (y-x)(z-x)(z-y).$$

14. Show that

$$\begin{vmatrix} 18 & 36 & 58 & 50 \\ 26 & 39 & 80 & 78 \\ 17 & 39 & 55 & 45 \\ 9 & 16 & 27 & 23 \end{vmatrix} = \begin{vmatrix} 0 & 4 & 4 & 4 \\ -1 & -9 & -1 & 9 \\ -1 & 7 & 1 & -1 \\ 9 & 16 & 27 & 23 \end{vmatrix}$$

15. Give two pairs of values of x and y which satisfy the equation

$$\begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 0.$$

16. Give the coördinates of two points on the line

$$\begin{vmatrix} x & y & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 0.$$

17. Trace the graph of

$$\begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 0.$$

18. Give the coördinates of two points on the line

$$\begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0.$$

19. Give three sets of values of x, y, z which satisfy the equation

$$\begin{vmatrix} x & y & z & 1 \\ 3 & 1 & -2 & 1 \\ 1 & -2 & 2 & 1 \\ -1 & 4 & 1 & 1 \end{vmatrix} = 0.$$

20. As in 19, for the equation

$$\begin{vmatrix} x & y & z & 1 \\ a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \end{vmatrix} = 0.$$

Prove the following identities:

21. $\cos(x + y) = \begin{vmatrix} \cos x & \sin x \\ \sin y & \cos y \end{vmatrix}.$

22. $\sin(x - y) = \begin{vmatrix} \sin x & \cos x \\ \sin y & \cos y \end{vmatrix}.$

23. $\cos 2x = \begin{vmatrix} \cos x & \sin x \\ \sin x & \cos x \end{vmatrix}.$

24. $$\begin{vmatrix} a & b & c \\ \sin x & \sin y & \sin z \\ \cos x & \cos y & \cos z \end{vmatrix} = a \sin(y - z) + b \sin(z - x) + c \sin(x - y).$$

25. $$\begin{vmatrix} \sin^2 x & \cos^2 x & 1 \\ \sin^2 y & \cos^2 y & 1 \\ \sin^2 z & \cos^2 z & 1 \end{vmatrix} = 0.$$

26. $$\begin{vmatrix} \cos x & \sin x \cos x & \cos x(\sin y + \sin z) \\ \cos y & \sin y \cos y & \cos y(\sin x + \sin z) \\ \cos z & \sin z \cos z & \cos z(\sin x + \sin y) \end{vmatrix} = 0.$$

27. $$\begin{vmatrix} \sin x & \sin 2x & \sin 3x \\ \sin^2 x & \sin^2 2x & \sin^2 3x \\ \sin 2x & \sin 4x & \sin 6x \end{vmatrix} = 2 \sin x \sin 2x \sin 3x (\sin 2x - 2 \sin x).$$

28. Show that

$$\begin{vmatrix} a + a' & b + b' & c + c' \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} + \begin{vmatrix} a' & b' & c' \\ d & e & f \\ g & h & k \end{vmatrix}$$

29. Show that the equations

$$-4x + y + z = 0 \quad \text{and} \quad x - 2y + z = 0$$

are satisfied by

$$x : y : z = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} : -\begin{vmatrix} -4 & 1 \\ 1 & 1 \end{vmatrix} : \begin{vmatrix} -4 & 1 \\ 1 & -2 \end{vmatrix}.$$

30. Show that the equations $\begin{cases} lx + my + nz = 0, \\ l'x + m'y + n'z = 0, \end{cases}$

are satisfied by

$$x : y : z = \begin{vmatrix} m & n \\ m' & n' \end{vmatrix} : -\begin{vmatrix} l & n \\ l' & n' \end{vmatrix} : \begin{vmatrix} l & m \\ l' & m' \end{vmatrix}.$$

31. Show that the equations $\begin{cases} 2x + 4y + 5z = 0, \\ 3x + 5y + 6z = 0, \\ 4x + 6y + 7z = 0, \end{cases}$ are satisfied by $x : y : z = 1 : -3 : 2$.

Solve the following systems of equations:

32. $$\begin{aligned} 2x + 3y - 4z + 7 &= 0, \\ 7x - 4y - 1 &= 0, \\ 9x - 4z + 1 &= 0. \end{aligned}$$

33. $$\begin{aligned} 20u + 2v - 7 &= 0, \\ 4v + 5w - 1 &= 0, \\ 4u - 3w + 2 &= 0. \end{aligned}$$

34. $$\begin{aligned} -r + s + t + u &= 4, \\ r - s + t + u &= 3, \\ r + s - t + u &= 2, \\ r + s + t - u &= 1. \end{aligned}$$

35. $$\begin{aligned} 2x - y - 3z + w &= 1, \\ x + 2y + z - w &= 2, \\ 3x - 3y - z + 2w &= -1, \\ -x - y + 2z - 3w &= 0. \end{aligned}$$

36. $$\begin{aligned} k + l + m - 2n &= 1, \\ 2k - l + 2m - 4n &= 2, \\ -k + 2l + 3m - 6n &= -2, \\ k - l + 4m - 8n &= -1. \end{aligned}$$

CHAPTER XII

PERMUTATIONS. COMBINATIONS. CHANCE

171. Permutations. — A *permutation* is a definite order or arrangement of a group of objects, or of part of the group.

Let there be a group of n distinct objects. The number of possible arrangements, taking r of these objects at a time, is called the *number of permutations of n things r at a time,* and is denoted by ${}_nP_r$.

Theorem 1. *The number of permutations of n things r at a time is*

$$ {}_nP_r = n(n-1)\cdots(n-r+1). $$

Proof. Evidently $\qquad {}_nP_1 = n.$

Now with each of the n objects we may pair any one of the remaining $n - 1$ objects.

Hence $\qquad {}_nP_2 = n(n-1).$

With each one of these $n(n-1)$ permutations containing 2 objects we may associate one of the remaining $n - 2$ objects.

Hence $\qquad {}_nP_3 = n(n-1)(n-2).$

Proceeding in this way we obtain the formula stated.

When $\qquad r = n$

we have $\qquad {}_nP_n = n!.$

Exercises.

1. How many numbers of four figures each can be formed from the digits 1, 2, 3, 4?
2. How many 3-figure numbers can be formed from the digits 1, 2, 3, 4, 5?
3. How many numbers greater than 1000 can be formed from the digits 1, 3, 5, 7, 9?
4. How many changes can be rung with 8 bells, 4 at a time?

172. Combinations. — A combination is a group of objects, without reference to their arrangement.

The number of different groups or combinations of n objects, each group containing r objects, is called the number of combinations of n things r at a time, and is denoted by ${}_nC_r$.

Theorem 2. *The number of combinations of n things r at a time is*

$$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n(n-1)\cdots(n-r+1)}{r!}. $$

Proof. Suppose all the combinations of the n things r at a time to be written down. Each group so written will yield, by permuting its objects in all possible ways, $r!$ permutations. Hence there are $r!$ times as many permutations as combinations, or

$$ r!\,{}_nC_r = {}_nP_r = n(n-1)\cdots(n-r+1). $$

Hence the theorem.

See also the definition of ${}_nC_r$ as a binomial coefficient in (**52**).

Exercises.

1. How many triangles can be formed from 6 points, no three points being collinear?

2. How many tetrahedrons can be formed from 12 points, no four points being coplanar?

3. How many committees of 3 persons each can be formed from a club of 10 persons?

4. Show that ${}_nC_r = {}_nC_{n-r}$.

(This is a convenient formula when r is nearly as large as n. It is then shorter to calculate ${}_nC_{n-r}$.)

5. Show that ${}_nC_0 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_n = 2^n$.

(Expand $(1+x)^n$ and put $x = 1$; ${}_nC_0$ is *defined* to be 1.)

6. How many committees, consisting of from 1 to 9 members, can be formed from a club of 10 persons?

7. Find the value of ${}_{20}C_{18}$.

173. Theorem 3. — *The number of permutations of n things n at a time, when p things are alike, is*

$$ P = \frac{n!}{p!}. $$

Proof. Let P be the number of permutations sought, and suppose them written down. If now the p things in question were unlike, by permuting them among themselves each of the P permutations would yield $p!$ permutations; the total number of permutations so formed would be $p!\,P$ and must equal ${}_nP_n$ or $n!$. Hence the theorem.

Similarly, the number of permutations of n things n at a time, when p things are all of one kind, and q of a second kind, will be

$$P = \frac{n!}{p!q!},$$

and so on.

174. Exercises.

1. How many permutations of seven letters each can be formed from the letters of the word "arrange"?
2. How many permutations of 11 letters each can be formed from the letters of the word "Mississippi"?
3. How many words, each containing a vowel and two consonants, can be formed from 4 vowels and 6 consonants?
4. How many even numbers of four figures each can be formed from the digits 1, 2, 3, 4, 5, 6?
5. How many elevens can be chosen from 20 players if only 6 of the 20 are qualified to play behind the line?
6. As in 5, if in addition, only 2 men are qualified for center.
7. How many sums can be formed with one coin of each denomination, from a cent to a dollar?
8. As in 7, except that there are two coins of each denomination.
9. If two coins are tossed, in how many ways may they fall?
10. As in 9, for 10 coins.
11. If two dice are thrown, in how many ways may they turn up?
12. As in 11, for 3 dice.

175. Probability or Chance. — If a bag contains 4 white and 3 black balls, and a ball is drawn at random, what is the chance that it be white?

In order to solve this problem we first define chance or probability.

Definition. The measure of the probability of the occurrence of an event is taken to be the quotient:

$$\frac{\textbf{number of favorable ways}}{\textbf{total number of possible ways}}.$$

In the problem above, since there are 7 balls altogether, there are 7 possible ways of drawing one ball; of these 4 are favorable, since there are 4 white balls. Hence the chance that a white ball be drawn is $\frac{4}{7}$.

Similarly the chance for a black ball is $\frac{3}{7}$.

If an event can happen in a ways, and fail in b ways, then, by the definition, the chance that it will happen is $\frac{a}{a+b}$, and that it will fail is $\frac{b}{a+b}$.

Since the event must either happen or fail, the probability for which is $\frac{a}{a+b} + \frac{b}{a+b} = 1$, we have 1 as the mathematical symbol for certainty.

If p is the probability that an event will happen, $1 - p$ is the probability that it will fail.

Example 1. From a bag containing 4 white and 3 black balls, 2 balls are drawn at random.

(*a*) What is the chance that both be white?
Number of favorable ways: ${}_4C_2 = 6$.
Number of possible ways: ${}_7C_2 = 21$.
Hence the required chance is: $p = \frac{6}{21} = \frac{2}{7}$.

(*b*) What is the chance that at least one be white?
Favorable cases: both white, ${}_4C_2 = 6$;
one white, other black, $3 \times 4 = 12$.

∴ Total number of favorable cases is 18.
Number of possible cases, as before, 21.

Hence $p = \frac{18}{21} = \frac{6}{7}$.

A shorter method is as follows: The probability that both balls be black is $\frac{{}_3C_2}{{}_7C_2} = \frac{3}{21} = \frac{1}{7}$. Hence the chance that at least one be white is $1 - \frac{1}{7} = \frac{6}{7}$.

Example 2. From 12 tickets, numbered 1, 2, · · · 12, four are drawn at random.

(*a*) What is the probability that they bear even numbers?

Since 6 tickets bear even numbers, the number of favorable cases is ${}_6C_4$. The total number of ways of drawing 4 tickets from 12 is ${}_{12}C_4$. Hence

$$p = \frac{{}_6C_4}{{}_{12}C_4} = \frac{6 \cdot 5 \cdot 4 \cdot 3}{12 \cdot 11 \cdot 10 \cdot 9} = \frac{1}{33}.$$

(*b*) What is the chance that two bear even, the other odd numbers?

We can select two tickets bearing even numbers in ${}_6C_2$ ways; also two bearing odd numbers in ${}_6C_2$ ways. Combining any one of the first with any one of the second gives ${}_6C_2 \times {}_6C_2$ favorable ways. Hence

$$p = \frac{{}_6C_2 \times {}_6C_2}{{}_{12}C_4} = \frac{5}{11}.$$

176. Exercises.

1. If 5 coins are tossed, what is the chance of three heads?

2. If 5 coins are tossed, what is the chance of at least two heads?

3. If 3 balls are drawn from a bag containing 5 white and 4 black balls, what is the chance that all three are white?

4. In exercise 3, what is the chance of drawing 2 white balls and one black ball?

5. In exercise 3, what is the chance of drawing at least one white ball?

6. What is the chance of two sixes in a single throw of two dice?

7. What is the chance of throwing three sixes in a single throw with three dice?

8. Three dice are thrown. What is the chance that the sum of the numbers turned up is 11?

9. As in 8, except that the sum is to be 7.

10. Six cards are drawn from a pack of 52. What is the chance of three aces?

11. Six cards are drawn from a pack of 52. What is the chance that all are of the same suite?

177. Compound Probabilities.

Definition. Two events are said to be independent when the occurrence of one does not affect that of the other.

Theorem 4. *The chance that both of two independent events shall happen is the product of their separate probabilities.*

Proof. Suppose the first event happens in a ways and fails in b ways, out of $a + b$ possible ways, and that the second happens in a' ways and fails in b' ways, out of a total of $a' + b'$ ways.

Combining each of the a favorable ways of the first event with each of the a' favorable ways of the second, we have aa' favorable cases. The total number of possible cases is $(a + b)(a' + b')$. Hence

$$p = \frac{aa'}{(a+b)(a'+b')} = \frac{a}{a+b} \times \frac{a'}{a'+b'},$$

which is the product of the separate probabilities of the two events.

As an immediate extension, we have

Theorem 5. *If the probabilities of several independent events be* $p_1, p_2 \cdots p_n$, *the probability that all will happen is*

$$P = p_1 \times p_2 \times \cdots \times p_n.$$

Example. From a bag containing 4 white and 3 black balls, 2 balls are drawn in succession. What is the chance that both are white?

On the first drawing the chance for a white ball is $\frac{4}{7}$; on the second, $\frac{3}{6}$. The probability of both events is therefore

$$\tfrac{4}{7} \times \tfrac{3}{6} = \tfrac{2}{7}.$$

Definition. Two events are said to be dependent when the occurrence of one of them affects that of the other.

Theorem 6. *Of n dependent events, let the chance that the first will happen be p_1, the chance that the second then follows be p_2, that the third then follows be p_3, and so on. The chance that all these events shall happen is then*

$$P = p_1 \times p_2 \times p_3 \times \cdots p_n.$$

This is an immediate consequence of the preceding theorem.

Theorem 7. *If p be the chance that an event will happen in one trial, the chance that it will happen just r times in n trials is*

$$P = {}_nC_r p^r (1-p)^{n-r}.$$

Proof. The chance that r trials out of n shall succeed is p^r, and that the other $n-r$ trials shall fail is $(1-p)^{n-r}$. Hence the probability of success in r particular trials and of failure in the $n-r$ other trials is $p^r(1-p)^{n-r}$. But of the n trials, any r may be the successful ones, which gives ${}_nC_r$ possibilities, each having a probability $p^r(1-p)^{n-r}$. Hence the result stated.

1. In a class of 3 students, A solves on the average 9 problems out of 10, B 8 out of 10, C 7 out of 10. What is the chance that a problem, presented to the class, will be solved?

The problem will be solved unless all three students fail, the probability for which is

$$\tfrac{1}{10} \times \tfrac{2}{10} \times \tfrac{3}{10} = \tfrac{3}{500}.$$

Hence the chance that the problem will be solved is

$$1 - \tfrac{3}{500} = \tfrac{497}{500}.$$

2. Two bags each contain 5 black balls, and a third bag contains 5 black and 5 white balls. What is the chance of drawing a white ball from one of the bags selected at random?

The chance that the bag containing white balls be chosen is $\frac{1}{3}$. The chance that a white ball be now drawn from this bag is $\frac{1}{2}$. Hence the probability that both events happen and that a white ball be drawn is

$$\tfrac{1}{3} \times \tfrac{1}{2} = \tfrac{1}{6}.$$

3. A coin is tossed 10 times. What is the chance for just 3 heads?

The probability of a head in one trial is $\frac{1}{2}$. Hence

$$ {}_nC_r p^r (1-p)^{n-r} = {}_{10}C_3 \left(\tfrac{1}{2}\right)^3 \left(1-\tfrac{1}{2}\right)^7 = \tfrac{15}{128}. $$

178. Exercises.

1. Three hats each contain 5 tickets, those in two of the hats being numbered 1, 2, · · · 5, and those in the third hat being blank. What is the chance of drawing a ticket bearing an even number from one of the hats selected at random?

2. If in exercise 1 two tickets be drawn from a hat chosen at random, what is the chance that both bear even numbers?

3. If each of two persons draw a ticket from one of the hats in exercise 1, the first ticket being replaced before the second is drawn, what is the chance that both persons draw the same number? What is the chance that both draw blanks?

4. If a coin be tossed 10 times, what is the chance for at least 7 heads?

5. How many different sets of throws can be made with a coin, each set consisting of 5 successive throws?

6. The chance that a person aged 25 years will live to be 75 is $\frac{7}{24}$. What is the chance that, of three couples married at the age of 25, at least one shall live to celebrate their golden wedding?

7. A bag contains 10 white, 6 black, and 4 red balls. Find the chance that, of three balls drawn, there shall be one of each color.

8. A gunner hits the target on an average 7 times out of 10. What is the chance that 5 consecutive shots shall hit the target?

9. Two dice are thrown. Find the chance that the sum of the numbers turned up shall be even.

CHAPTER XIII

THEORY OF EQUATIONS

179. Standard Equation of nth Degree. — This is the equation

$$(1) \qquad p_0x^n + p_1x^{n-1} + p_2x^{n-2} \cdots + p_{n-1}x + p_n = 0;$$

p_0x^n is called the **leading term** and p_n the **constant** (or **absolute**) **term.**

The coefficient of the leading term may be made equal to unity by dividing the whole equation by this coefficient.

When all the terms written in equation (1) are present, the equation is said to be **complete;** when one or more terms are absent, the equation is said to be **incomplete.** An incomplete equation may be made complete by supplying the missing terms with zero coefficients.

We shall represent the polynomial forming the left member of equation (1) by $f(x)$; $f(a)$ shall denote the value of this polynomial when $x = a$, $f(b)$ the value when $x = b$, and so on.

A **root** of an equation is a value of x which satisfies the equation; hence a is a root of the equation $f(x) = 0$ if $f(a) = 0$.

In the present chapter we shall consider methods of finding the roots of the equation $f(x) = 0$.

180. Factor Theorem. — *If a is a root of the equation $f(x) = 0$, then $f(x)$ is divisible by $(x - a)$, and conversely.*

Proof. Divide $f(x)$ by $(x - a)$; let Q be the quotient, R the remainder. Then

$$f(x) = (x - a)Q + R.$$

Putting $x = a$, we obtain $R = 0$, since $f(a) = 0$ by hypothesis. Hence $f(x)$ is divisible by $(x - a)$ without a remainder.

Conversely, assume

$$f(x) = (x - a)Q.$$

Put $x = a$ and we have $f(a) = 0$, since Q is finite when $x = a$; hence a is a root of $f(x) = 0$. [See also **(11)**, (*f*).]

181. Depressed Equation. — When a is a root of the equation $f(x) = 0$, we may write

$$f(x) = (x - a)\,Q.$$

Any other value of x which reduces $f(x)$ to zero must also reduce Q to zero, and is therefore a root of the equation $Q = 0$.

But if $f(x)$ is of degree n, Q will be of degree $n - 1$. Hence when one root of an equation is known, the other roots may be found by solving an equation of degree one less than that of the given equation, and whose left member is found by dividing the left member of the given equation by (x — *the root*).

The new equation is called the *depressed equation.*

Exercises. Show that each of the following equations has the root indicated, and find the other roots:

1. $x^3 - 9x^2 + 26x - 24 = 0$; $x = 2$.
2. $x^3 + 3x^2 - 8x - 24 = 0$; $x = -3$.
3. $3x^3 - 14x^2 + 17x - 6 = 0$; $x = \frac{2}{3}$.

182. Number of Roots. — We *assume* that every equation of the form (1), (**179**), has at least one root, that is, there exists at least one value of x, real or imaginary, which satisfies the equation. It can then be shown that *an equation of the nth degree has just n roots.*

For, let a_1 be a root. Form the depressed equation by removing from $f(x)$ the factor $x - a_1$, and let the new equation, of degree $n - 1$, be $f_1(x) = 0$. By the above assumption, this equation has a root, say a_2. Removing from $f_1(x)$ the factor $x - a_2$, we obtain a new equation, $f_2(x) = 0$, of degree $n - 2$, and so on. After $n - 1$ steps, by which $n - 1$ roots are removed, we have an equation of the first degree which gives one more root. Hence there are just n roots.

183. To Form an Equation Having Given Roots. — Let it be required to form an equation whose roots are $a_1, a_2, a_3, \cdots a_n$.

Obviously the required equation is

$$A\,(x - a_1)(x - a_2)(x - a_3) \cdots (x - a_n) = 0,$$

A being an arbitrary constant.

Exercises. Form the equations whose roots are:

1. 1, 2, 3.
2. 1, −1, 2.
3. 2, 2, −2, 0.
4. −1, −2, −3, −4.
5. $\pm 1, \frac{1}{2}, \frac{1}{3}$.
6. $-\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$.

(Write the results from exercises 5 and 6 with integral coefficients.)

184. Relations between Coefficients and Roots. — In the case of 2, 3, and 4 roots respectively we find on expanding,

$$(x - a_1)(x - a_2) = x^2 - (a_1 + a_2)x + a_1a_2.$$

$$(x - a_1)(x - a_2)(x - a_3) = x^3 - (a_1 + a_2 + a_3)x^2 + (a_1a_2 + a_2a_3 + a_1a_3)x - a_1a_2a_3.$$

$$(x-a_1)(x-a_2)(x-a_3)(x-a_4) = x^4 - (a_1 + a_2 + a_3 + a_4)x^3 + (\cdot\cdot\cdot)x^2 - (\cdot\cdot\cdot)x + a_1a_2a_3a_4.$$

We here observe three facts, namely:

1. *The coefficient of the leading term is unity;*
2. *The coefficient of the second term is the negative sum of the roots;*
3. *The constant term is the product of the roots, plus when the number of roots is even, minus when odd.*

We shall show by induction that these results are true in general.

Assume them to be true for $n - 1$ roots; then if the equation whose roots are $a_1, a_2, \cdots a_{n-1}$, be

$$x^{n-1} + q_1x^{n-2} + \cdots + q_{n-1} = 0,$$

we have by hypothesis,

$$q_1 = -(a_1 + a_2 + \cdots + a_{n-1}); \quad q_{n-1} = (-1)^{n-1}a_1a_2 \cdots a_{n-1}.$$

Multiplying the above equation by $(x - a_n)$, which introduces the new root a_n, we find on collecting in powers of x:

$$x^n + (q_1 - a_n)x^{n-1} + \cdots - a_nq_{n-1} = 0,$$

or; $$x^n - (a_1 + a_2 + \cdots + a_{n-1} + a_n)x^{n-1} + \cdots + (-1)^n a_1a_2 \cdots a_{n-1}a_n = 0.$$

Hence if the results are true for the case of $n - 1$ roots, they hold also for n roots. But they are true for 4 roots, hence also for 5, hence for 6, and so on.

Exercise. Show by induction that the coefficient of the third highest power of x equals the sum of the products of the roots taken two at a time.

185. Fractional Roots. — *An equation with integral coefficients, in which the coefficient of the leading term is unity, cannot have as a root a rational fraction in its lowest terms.*

Proof. Assume that the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0$$

has integral coefficients and that one of its roots is $\frac{a}{b}$, where a and b are integers, relatively prime. Putting $x = \frac{a}{b}$ we have,

$$\left(\frac{a}{b}\right)^n + p_1\left(\frac{a}{b}\right)^{n-1} + p_2\left(\frac{a}{b}\right)^{n-2} + \cdots + p_{n-1}\left(\frac{a}{b}\right) + p_n = 0.$$

Multiplying through by b^{n-1} and transposing,

$$\frac{a^n}{b} = -\left(p_1a^{n-1} + p_2a^{n-2}b + \cdots + p_{n-1}ab^{n-2} + p_nb^{n-1}\right).$$

Here we have a fraction in its lowest terms equal to an integer, which is impossible. Hence $\frac{a}{b}$ cannot be a root.

Corollary. Any rational root of an equation whose coefficients are integers and whose leading coefficient is unity must be an integer.

186. Imaginary Roots. — *If the general equation of nth degree, with real coefficients, has an imaginary root $a + ib$, then also the conjugate imaginary, $a - ib$, is a root.*

Proof. Assume that $a + ib$ is a root of the equation

$$f(x) \equiv x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0.$$

Then

$$(a + ib)^n + p_1(a + ib)^{n-1} + p_2(a + ib)^{n-2} + \cdots + p_{n-1}(a + ib) + p_n = 0.$$

Expanding the binomials, reducing all powers of i to ± 1 or $\pm i$, and collecting terms, we have a result of the form

$$f(a + ib) = P + iQ = 0.$$

Hence by Corollary 2 in (**36**),

$$P = 0 \quad \text{and} \quad Q = 0.$$

Now substitute $a - ib$ for x and proceed as before. The result will be

$$f(a - ib) = P - iQ,$$

since the only difference is in the sign of i. But $P = 0$ and $Q = 0$, hence $P - iQ = 0$, or $f(a - ib) = 0$. Therefore $a - ib$ is a root.

We may state our result as follows: *Imaginary roots, if present at all, always occur in conjugate pairs.*

187. Transformation of Equations. — In the following discussion we assume that any missing powers of x are inserted, supplied with zero coefficients, so as to make the equation formally complete. We consider the equation

$$f(x) \equiv p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0.$$

I. To change the signs of the roots.

Put $x = -y$. We obtain,

$$p_0(-y)^n + p_1(-y)^{n-1} + p_2(-y)^{n-2} + \cdots + p_{n-1}(-y) + p_n = 0,$$

or, on multiplying through by $(-1)^n$,

$$p_0y^n - p_1y^{n-1} + p_2y^{n-2} - \cdots + (-1)^{n-1}p_{n-1}y + (-1)^np_n = 0.$$

Hence, *to change the signs of the roots, change the signs of alternate coefficients, beginning with the second term.*

II. To multiply the roots by a constant factor, *m*.

Replace x by $\frac{y}{m}$ (so that $y = mx$). Then

$$p_0\left(\frac{y}{m}\right)^n + p_1\left(\frac{y}{m}\right)^{n-1} + p_2\left(\frac{y}{m}\right)^{n-2} + \cdots + p_{n-1}\left(\frac{y}{m}\right) + p_n = 0.$$

Multiplying through by m^n, we have,

$$p_0y^n + mp_1y^{n-1} + m^2p_2y^{n-2} + \cdots + m^{n-1}p_{n-1}y + m^np_n = 0.$$

Hence, *to multiply the roots by a constant factor m, multiply the coefficients in order, beginning with the second, by m, m^2, m^3, $\cdots$ m^n.*

When $m = -1$ we obtain the preceding rule for changing the signs of the roots.

III. To increase the roots by a constant quantity, *h*.

Replace x by $y - h$ (so that $y = x + h$). Then

$$p_0(y-h)^n + p_1(y-h)^{n-1} + p_2(y-h)^{n-2} + \cdots$$
$$+ p_{n-1}(y-h) + p_n = 0.$$

Expanding the binomials and collecting in powers of y, we obtain a result of the form,

$$p_0y^n + P_1y^{n-1} + P_2y^{n-2} + \cdots + P_{n-1}y + P_n = 0.$$

We shall now show how to obtain the coefficients $P_1, P_2, \cdots P_n$.

Replacing y in the last equation by $x + h$, the result must be the original equation, $f(x) = 0$. Hence

$$f(x) = p_0 (x + h)^n + P_1 (x + h)^{n-1} + P_2 (x + h)^{n-2} + \cdots \\ + P_{n-1} (x + h) + P_n.$$

This shows that if $f(x)$ be divided by $x + h$, the remainder is P_n. If the quotient be divided by $x + h$, the remainder is P_{n-1}; dividing the second quotient by $(x + h)$, the remainder is P_{n-2}, and so on.

Hence, *to increase the roots of the equation by h, divide $f(x)$ by $x + h$, then divide the quotient by $x + h$, divide the new quotient by $x + h$, and so on. The successive remainders are, in order,*

$$P_n, P_{n-1}, P_{n-2}, \cdots, P_1.$$

A concise method for performing the required divisions will be explained in the next article.

188. Synthetic Division. — When h and the coefficients p_0, p_1, p_2, $\cdots$ p_n are integers, the work of dividing $f(x)$ may be performed by the method of *synthetic division.* We shall illustrate this by increasing the roots of the equation

$$x^3 - 8x - 15 = 0$$

by 2.

Performing the first division at length, we have:

$$\begin{array}{rrrr|l}
x^3 + 0x^2 & - 8x & - 15 & & x + 2 \\
\underline{x^3 + 2x^2} & & & & \overline{x^2 - 2x - 4} \text{ quotient.} \\
-2x^2 & - 8x & & & \\
\underline{-2x^2} & \underline{- 4x} & & & \\
 & -4x & - 15 & & \\
 & \underline{-4x} & \underline{- 8} & & \\
 & & - 7 & \text{remainder.} &
\end{array}$$

We first shorten this operation by omitting to write the powers of x, using only the *detached coefficients, thus*:

$$\begin{array}{rrr|l}
1 + 0 & - 8 & - 15 & 1 + 2 \\
\underline{1 + 2} & & & \overline{1 - 2 - 4} \\
-2 & - 8 & & \\
\underline{-2} & \underline{- 4} & & \\
 & -4 & - 15 & \\
 & \underline{-4} & \underline{- 8} & \\
 & & - 7 &
\end{array}$$

This may be written more compactly as follows:

$$\begin{array}{ll}
& 1 + 0 - 8 - 15 \;\underline{|+2} \\
& \quad\; +2 - 4 - \;\; 8 \\ \cline{2-2}
\textit{1st quotient} & 1 - 2 - 4 \mid -7 \ \textit{remainder.}
\end{array}$$

Dividing the quotient by $x + 2$ we have,

$$\begin{array}{ll}
& 1 - 2 - 4 \;\underline{|+2} \\
& \quad\; +2 - 8 \\ \cline{2-2}
\textit{2nd quotient} & 1 - 4 \mid +4 \quad \textit{remainder}
\end{array}$$

Dividing the second quotient by $x + 2$ we have,

$$\begin{array}{ll}
& 1 - 4 \;\underline{|+2} \\
& \quad\; +2 \\ \cline{2-2}
\textit{3rd quotient} & 1 \mid -6 \quad \textit{remainder}
\end{array}$$

The whole operation may now be written thus:

$$\begin{array}{ll}
1 + 0 - 8 - 15 \;\underline{|+2} & \\
\quad\; +2 - 4 - \;\; 8 & \\ \hline
1 - 2 - 4 \mid - \;\; 7 & \textit{1st remainder} \\
\quad\; +2 - 8 \mid & \\
1 - 4 \mid +4 & \textit{2nd remainder} \\
\quad\; +2 \mid & \\
\underline{\underline{1}} \mid -6 & \textit{3rd remainder}
\end{array}$$

Then the transformed equation is:

$$x^3 - 6x^2 + 4x - 7 = 0.$$

To diminish the roots of an equation by h, proceed as above with $x - h$ in place of $x + h$. As an example, we diminish by 4 the roots of the equation

$$x^4 - 5x^3 + 7x^2 - 17x + 11 = 0.$$

$$\begin{array}{ll}
1 - \;\; 5 + \;\; 7 - 17 + 11 \;\underline{|-4} & \\
\quad - \;\; 4 + \;\; 4 - 12 + 20 & \\ \hline
1 - \;\; 1 + \;\; 3 - \;\; 5 \mid - \;\; 9 & \textit{1st remainder} \\
\quad - \;\; 4 - 12 - 60 \mid & \\
1 + \;\; 3 + 15 \mid +55 & \textit{2nd remainder} \\
\quad - \;\; 4 - 28 \mid & \\
1 + \;\; 7 \mid +43 & \textit{3rd remainder} \\
\quad - \;\; 4 \mid & \\
\underline{\underline{1}} \mid +11 & \textit{4th remainder}
\end{array}$$

Hence the transformed equation is:

$$x^4 + 11x^3 + 43x^2 + 55x - 9 = 0.$$

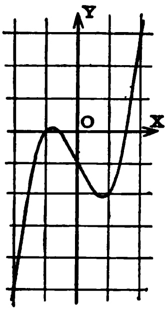
In using the method of synthetic division note that the coefficient of the leading term remains unchanged.

189. The graph of the equation $y = f(x)$, when

$$f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n.$$

To construct the graph which shall represent the fluctuating values of y as x varies, we assume a series of numerical values for x, calculate the corresponding values of y, and plot the points (x, y). On drawing a smooth curve through these points, we obtain a graph such as that in the figure, which represents the equation

$$y = x^3 - 2x - 1.$$



$y = x^3 - 2x - 1$

Here a set of corresponding values of x and y are:

$$x = \quad 0, \quad 1, \quad 2, \cdots, -1, \quad -2, \cdots;$$
$$y = -1, -2, \quad 3, \cdots, \quad 0, \quad -5, \cdots.$$

Since the curve crosses the x-axis when $y = 0$, we see that *the abscissas of the points where the graph of the equation $y = f(x)$ crosses the x-axis (called the x-intercepts of the graph) are the real roots of the equation $f(x) = 0$.*

An inspection of the above graph shows that one root of the equation $x^3 - 2x - 1 = 0$ is -1, another root lies between -1 and 0, and the third between $+1$ and $+2$. On removing the factor $x + 1$ from this equation, the depressed equation is $x^2 - x - 1 = 0$. Hence the exact values of the other two roots are $\frac{1}{2}(1 \pm \sqrt{5})$, or approximately, $+1.62$ and -0.62.

190. Effect of Changing the Constant Term. — Suppose that we add a quantity k to the constant term of $f(x)$, so that the equation

$$y = f(x)$$

becomes

$$y = f(x) + k.$$

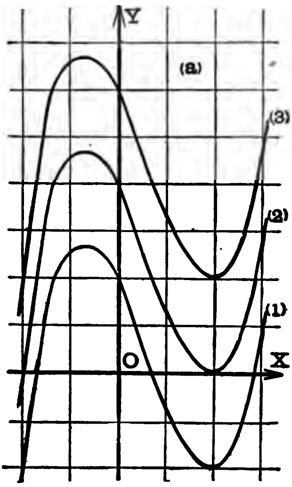
Suppose the curve $y = f(x)$ to be plotted; on adding k to each of its ordinates, we obtain the graph of $y = f(x) + k$. That is, *if k be added to the constant term of the equation $y = f(x)$, the graph is displaced vertically through the distance k, upward if k is plus, downward if k is minus.*

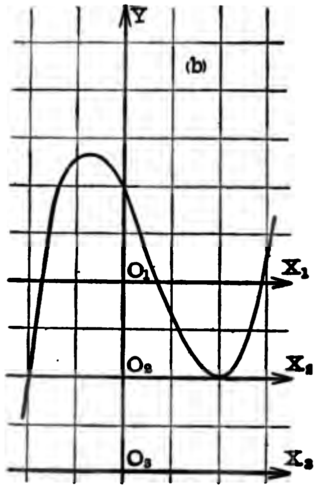
As an example, consider the equations

$$(1) \qquad y = \tfrac{1}{2}x^3 - x^2 - 2x + 2,$$

$$(2) \qquad y = \tfrac{1}{2}x^3 - x^2 - 2x + 4,$$

$$(3) \qquad y = \tfrac{1}{2}x^3 - x^2 - 2x + 6.$$





The graphs are shown in figure (*a*). The curves are of precisely the same form, but (2) lies two units higher than (1), and (3) two units higher than (2).

Instead of displacing the curve vertically, say upward, the same effect is produced in the graph by moving the x-axis an equal distance downward. Thus equations (1), (2), (3) are represented graphically by the curve in figure (*b*), y being counted from the lines O_1X_1, O_2X_2, O_3X_3 respectively.

191. Occurrence of Imaginary Roots in Pairs. — We can now consider article (**186**) geometrically. Thus in the first figure of (**190**), graph (1) shows that the equation

$$\tfrac{1}{2}x^3 - x^2 - 2x + 2 = 0$$

has three real unequal roots; replacing 2 by 4, the two positive roots become equal; that is, the equation

$$\tfrac{1}{2}x^3 - x^2 - 2x + 4 = 0$$

has three real roots, two of which are equal; finally on replacing 4 by 6, the two equal roots become imaginary; that is, the equation

$$\tfrac{1}{2}x^3 - x^2 - 2x + 6 = 0$$

has one real root and two imaginary roots.

In general, by changing the constant term in $f(x)$, the graph of $y = f(x)$ may be raised or lowered so that one of the "elbows" of the curve, which at first is cut by the x-axis, will become tangent to the x-axis, and on further changing the constant the x-axis will fail to intersect this elbow. Thus two real unequal roots first become equal, then imaginary.

192. Exercises. — Multiply the roots of the equation

1. $x^3 + x^2 - x - 1 = 0$ by 2;
2. $x^3 - 2x + 1 = 0$ by -2;
3. $x^3 - 48x - 112 = 0$ by $\tfrac{1}{2}$;
4. $x^4 + 6x^3 + 3x^2 - 26x - 24 = 0$ by $-\tfrac{1}{2}$.

Multiply the roots of the following equations by the smallest factor which will make all coefficients integers.

5. $x^2 + x + \tfrac{1}{4} = 0.$
6. $\tfrac{1}{2}x^3 - x^2 + \tfrac{1}{32} = 0.$
7. $x^2 - \tfrac{1}{2}x - \tfrac{1}{8} = 0.$
8. $x^3 - .1x^2 + .01x = 0.$
9. $x^3 + \tfrac{1}{9}x^2 - \tfrac{1}{81} = 0.$
10. $x^4 + 1.2x^2 - .225x + .015 = 0.$

Increase the roots of the equation

11. $x^3 - 3x^2 + 4 = 0$ by 2.
12. $4x^3 - 3x - 1 = 0$ by 3.
13. $x^4 - 2x^3 - 11x^2 + 12x + 36 = 0$ by -2.
14. $x^4 - 2x^3 - 39x^2 + 40x + 400 = 0$ by -4.

In the following equations increase the roots by a quantity such that the term involving the second highest power of x shall disappear.

15. $x^3 - 3x^2 + 2 = 0.$

16. $x^3 - 2x^2 + 1 = 0.$

17. $x^3 - 3x^2 - 6x + 1 = 0.$

18. $x^4 - 4x^3 - 8x + 32 = 0.$

In the following equations change the constant so that two roots shall become imaginary.

19. $x^3 - x^2 - 2x = 0.$

20. $x^3 - 3x^2 + 3 = 0.$

21. $x^3 - 3x - 2 = 0.$

22. $x^3 - x^2 - x + 1 = 0.$

Solve the following equations, given one root.

23. $x^3 - 2x^2 + x - 2 = 0; \quad x = \sqrt{-1}.$

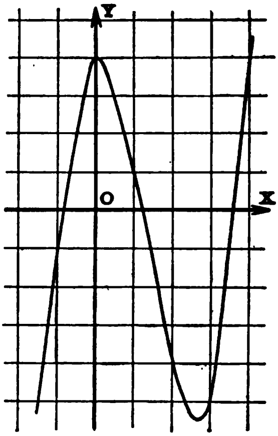
24. $2x^4 - 3x^3 + 5x^2 - 6x + 2 = 0; \quad x = -2\sqrt{-1}.$

25. $x^5 - 8x^3 - 8x^2 + 64 = 0; \quad x = -1 - \sqrt{-3}.$

193. Approximation to the Roots of an Equation. — In this article we shall illustrate a method for obtaining, to any desired degree of accuracy, any real root of an algebraic equation. As an example we shall obtain, to four decimal places inclusive, one of the roots of the equation

$$(1) \qquad f(x) \equiv x^3 - 4x^2 + 4 + 0.$$

The graph is given in the figure.



First Step. Location of Real Roots. We first locate the real roots by trial. As a set of corresponding values of x and $f(x)$ we have

$$x = -2, \quad -1, \quad 0, \quad +1, \quad +2, \quad +3, \quad +4.$$
$$f(x) = -20, \quad -1, \quad +4, \quad +1, \quad -4, \quad -5, \quad +4.$$

When $f(x)$ changes sign, the graph crosses the x-axis, and at least one root must lie between the corresponding values of x. Hence there is a root between -1 and 0, another between $+1$ and $+2$, and a third between $+3$ and $+4$. But there cannot be more than three roots, since a cubic expression cannot contain more than three linear factors. Hence there is just one root in each of the above intervals.

We shall proceed to obtain the root between 1 and 2.

Second Step. Diminish the roots of the given equation by the integral part of the root required (**187**)

$$\begin{array}{lllll}
1 & -\,4 & +\,0 & +\,4 \,\rvert & \underline{-\,1} \\
 & -\,1 & +\,3 & +\,3 & \\ \hline
1 & -\,3 & -\,3 \,\rvert & +\,1 & \\
 & -\,1 & +\,2 \,\rvert & & \\ \hline
1 & -\,2 \,\rvert & -\,5 & & \\
 & -\,1 \,\rvert & & & \\ \hline
1 \,\rvert & -\,1 & & &
\end{array}$$

The transformed equation is

(2) $$x^3 - x^2 - 5x + 1 = 0.$$

Since (1) has a root between 1 and 2, (2) must have a root between 0 and 1, that is, a decimal root. To make this root an integer, we take the

Third Step. Multiply the roots of the transformed equation by 10 (**187**).

The new equation is

(3) $$x^3 - 10x^2 - 500x + 1000 = 0.$$

The root of (3) between 0 and 10 will give the first decimal of the required root of (1). If we neglect the terms in x^3 and x^2 in (3) we get an approximate value, $x = 2$. Putting $x = 2$ in (3), the left member is negative; now putting $x = 1$, the left member is positive. Hence the root lies between 1 and 2, and the required root of (1) is 1.1+.

We now repeat these steps and obtain the first decimal of the root of (3), which will be the second decimal of the root of (1), and so on. Indicating the three steps in order by (*a*), (*b*), (*c*), we obtain the successive decimals of the root as shown below, the process of finding the first decimal being included for completeness.

(3) $$x^3 - 10x^2 - 500x + 1000 = 0.$$

(*a*) *Locate the root between* 0 *and* 10.

Neglect terms in x^3 and x^2; then $x = 2$. Try this value and the next smaller value (or larger if the left member of (3) does not change sign) and the root is located between 1 and 2.

(*b*) *Diminish roots by figure found in* (*a*).

$$\begin{array}{rrrr|l}
1 & -\ 10 & -\ 500 & +\ 1000 & \underline{-\ 1} \\
 & -\ 1 & +\ 9 & +\ 509 & \\ \cline{1-4}
1 & -\ 9 & -\ 509 & +\ \mathbf{491} & \\
 & -\ 1 & +\ 8 & & \\ \cline{1-3}
1 & -\ 8 & -\ \mathbf{517} & & \\
 & -\ 1 & & & \\ \cline{1-2}
1 & -\ \mathbf{7} & & &
\end{array}$$

Transformed equation: $x^3 - 7x^2 - 517x + 491 = 0$.

(*c*) *Multiply roots by* 10.

$$x^3 - 70x^2 - 51{,}700x + 491{,}000 = 0.$$

Repeat these operations on the last equation.

(*a*) $$x = 491{,}000 \div 51{,}700 = 9+.$$

By trial the sign of the left member is + when x is 9 and 8, but changes when x is 10. Hence the root is between 9 and 10.

(*b*)

$$\begin{array}{rrrr|l}
1 & -\ 70 & -\ 51{,}700 & +\ 491{,}000 & \underline{-\ 9} \\
 & -\ 9 & +\ 549 & +\ 470{,}241 & \\ \cline{1-4}
1 & -\ 61 & -\ 52{,}249 & +\ \mathbf{20{,}759} & \\
 & -\ 9 & +\ 468 & & \\ \cline{1-3}
1 & -\ 52 & -\ \mathbf{52{,}717} & & \\
 & -\ 9 & & & \\ \cline{1-2}
1 & -\ \mathbf{43} & & &
\end{array}$$

$$x^3 - 43x^2 - 52{,}717x + 20{,}759 = 0.$$

(*c*) $$x^3 - 430x^2 - 5{,}271{,}700x + 20{,}759{,}000 = 0.$$

The required root of (1) is now $x = 1.19+$. Another repetition of the process gives the third decimal.

(*a*) $$x = 20{,}759{,}000 \div 5{,}271{,}700 = 4-.$$

The left member has opposite signs for $x = 3$ and $x = 4$, hence the root is between 3 and 4.

(*b*)

$$\begin{array}{rrrr|l}
1 & -\ 430 & -\ 5{,}271{,}700 & +\ 20{,}759{,}000 & \underline{-\ 3} \\
 & -\ 3 & +\ 1{,}281 & +\ 15{,}818{,}943 & \\ \cline{1-4}
1 & -\ 427 & -\ 5{,}272{,}981 & +\ \mathbf{4{,}940{,}057} & \\
 & -\ 3 & +\ 1{,}272 & & \\ \cline{1-3}
1 & -\ 424 & -\ \mathbf{5{,}274{,}253} & & \\
 & -\ 3 & & & \\ \cline{1-2}
1 & -\ \mathbf{421} & & &
\end{array}$$

$$x^3 - 421x^2 - 5{,}274{,}253x + 4{,}940{,}057 = 0.$$

We thus have the required root of (1) as $x = 1.193+$.

We may omit step (*c*) in our last operation and get the next figure of the required root by neglecting x^3 and x^2 in the last equation. This gives $x = .9+$, and our root is, finally,

$$x = 1.1939+.$$

A convenient arrangement of the whole operation of finding this root is as follows:

$$\begin{array}{rrrrl}
1 & -4 & +0 & +4 & \underline{|-1} \\
 & -1 & +3 & +3 & \\ \cline{1-4}
1 & -3 & -3 & \multicolumn{1}{|l}{+1} & \\
 & -1 & \multicolumn{1}{r|}{+2} & & \\ \cline{1-3}
1 & -2 & \multicolumn{1}{|l}{-5} & & \\
 & \multicolumn{1}{r|}{-1} & & & \\ \cline{1-2}
\underline{\underline{1}} & \multicolumn{1}{|l}{-1} & & &
\end{array}$$

$$\begin{array}{rrrrl}
1 & -10 & -500 & +1000 & \underline{|-1} \\
 & -1 & +9 & +509 & \\ \cline{1-4}
1 & -9 & -509 & \multicolumn{1}{|l}{+491} & \\
 & -1 & \multicolumn{1}{r|}{+8} & & \\ \cline{1-3}
1 & -8 & \multicolumn{1}{|l}{-517} & & \\
 & \multicolumn{1}{r|}{-1} & & & \\ \cline{1-2}
\underline{\underline{1}} & \multicolumn{1}{|l}{-7} & & &
\end{array}$$

$$\begin{array}{rrrrl}
1 & -70 & -51{,}700 & +491{,}000 & \underline{|-9} \\
 & -9 & +549 & +470{,}241 & \\ \cline{1-4}
1 & -61 & -52{,}249 & \multicolumn{1}{|l}{+20{,}759} & \\
 & -9 & \multicolumn{1}{r|}{+468} & & \\ \cline{1-3}
1 & -52 & \multicolumn{1}{|l}{-52{,}717} & & \\
 & \multicolumn{1}{r|}{-9} & & & \\ \cline{1-2}
\underline{\underline{1}} & \multicolumn{1}{|l}{-43} & & &
\end{array}$$

$$\begin{array}{rrrrl}
1 & -430 & -5{,}271{,}700 & +20{,}759{,}000 & \underline{|-3} \\
 & -3 & +1{,}281 & +15{,}818{,}943 & \\ \cline{1-4}
1 & -427 & -5{,}272{,}981 & \multicolumn{1}{|l}{+4{,}940{,}057} & \\
 & -3 & \multicolumn{1}{r|}{+1{,}272} & & \\ \cline{1-3}
1 & -424 & \multicolumn{1}{|l}{-5{,}274{,}253} & & \\
 & \multicolumn{1}{r|}{-3} & & & \\ \cline{1-2}
\underline{\underline{1}} & \multicolumn{1}{|l}{-421} & & &
\end{array}$$

Root, $1.1939+$.

194. In approximating to the roots of an equation, the following remarks should be borne in mind. Let the student supply proofs when needed.

(1) Every equation of odd degree has at least one *real* root. (For $f(x)$ has opposite signs when $x = +\infty$ and $x = -\infty$.)

(2) When an even number of roots lie between $x = a$ and $x = b$, $f(a)$ and $f(b)$ will have like signs.

(3) Whenever more than one root lies between two assumed values of x, especial care must be used to separate them by trial.

(4) The next decimal of a root is obtained approximately by dividing the absolute term of the last transformed equation by the coefficient of x with its sign changed.

(5) Should this decimal be too large, the constant term of the next transformed equation will change sign. (Observe that in the example the constant terms of the original equation and of all the transformed equations are of the same sign.)

(6) Should this decimal be too small, the next transformed equation will not have a root between 0 and 10, except when there happen to be two or more roots of the original equation with the same integral part.

(7) To obtain a negative root, change the signs of all the roots and proceed as for a positive root.

195. Exercises. — Calculate to four decimal places the real roots of the equations:

1. $x^3 - 24x - 48 = 0.$
2. $x^3 - 7x^2 + 4x + 24 = 0.$
3. $x^3 - 2x + 1 = 0.$
4. $x^3 - x^2 + x - 1 = 0.$
5. $x^3 + x^2 + x + 1 = 0.$
6. $x^4 - 6x^2 + 5 = 0.$
7. $x^3 - 7x - 5 = 0.$
8. $x^3 - 31x - 19 = 0.$
9. $x^3 - 48x - 112 = 0.$
10. $2x^3 - 18x^2 + 46x - 30 = 0.$
11. $7x^3 - 9x + 5 = 0.$
12. $4x^3 - 3x - 1 = 0.$
13. $x^4 + x^3 - 2x^2 - 3x - 3 = 0.$
14. $x^4 - 2x^3 - 8x^2 + 24x - 48 = 0.$
15. $x^4 - 4x^3 - 8x + 32 = 0.$
16. $x^4 + 2x^3 + x + 2 = 0.$
17. $3x^4 - 2x^3 - 16x^2 - 56x + 96 = 0.$
18. $x^3 - 7x - 7 = 0.$
19. $8x^4 + 16x^3 + 18x^2 + x + 7 = 0.$
20. $7x^3 + 8x^2 - 14x - 16 = 0.$
21. $2x^4 - 5x^3 - 32x + 80 = 0.$
22. $2x^5 - 4x^3 + 3x^2 - 6 = 0.$

196. Cardan's Solution of the Cubic Equation. — As in the case of the quadratic equation, so the equations of third and fourth degree may be solved by means of radicals. This cannot be done, in general, for equations of degree higher than the fourth. We give here a solution of the cubic equation

$$(1) \qquad a_0x^3 + 3\,a_1x^2 + 3\,a_2x + a_3 = 0.$$

We first obtain a new equation containing no term of second degree. To do this, put

$$x = y + h.$$

Expanding and collecting in powers of y,

$$a_0y^3 + 3\,(a_0h + a_1)\,y^2 + 3\,(a_0h^2 + 2\,a_1h + a_2)\,y$$
$$+ a_0h^3 + 3\,a_1h^2 + 3\,a_2h + a_3 = 0.$$

The term in y^2 drops out if

$$a_0h + a_1 = 0, \quad \text{or} \quad h = -\frac{a_1}{a_0}.$$

With this value of h the equation becomes

$$a_0y^3 + \frac{3\,(a_0a_2 - a_1^2)}{a_0}\,y + \frac{a_0^2a_3 - 3\,a_0a_1a_2 + 2\,a_1^3}{a_0^2} = 0.$$

Putting $$y = \frac{z}{a_0},$$

we have

$$z^3 + 3\,(a_0a_2 - a_1^2)\,z + (a_0^2a_3 - 3\,a_0a_1a_2 + 2\,a_1^3) = 0.$$

Let

$$H = a_0a_2 - a_1^2; \quad G = a_0^2a_3 - 3\,a_0a_1a_2 + 2\,a_1^3.$$

Then the equation becomes

$$(2) \qquad z^3 + 3\,Hz + G = 0.$$

To solve this equation let

$$z = \sqrt[3]{r} + \sqrt[3]{s}.$$

Then

$$z^3 = r + s + 3\,\sqrt[3]{rs}\,(\sqrt[3]{r} + \sqrt[3]{s}),$$

or, $$z^3 - 3\,\sqrt[3]{rs}\cdot z - (r + s) = 0.$$

If this is to be identical with (2), we must have

$$\sqrt[3]{rs} = -H, \quad \text{and} \quad r + s = -G;$$

or,

$$rs = -H^3, \quad \text{and} \quad r + s = -G.$$

Solving for r and s,

$$r = \frac{-G + \sqrt{G^2 + 4H^3}}{2}; \quad s = \frac{-G - \sqrt{G^2 + 4H^3}}{2}.$$

Then

$$z = \sqrt[3]{r} + \sqrt[3]{s} = \sqrt[3]{r} - \frac{H}{\sqrt[3]{r}}. \quad (rs = -H^3.)$$

Let the three cube roots of r be α_1, α_2, and α_3. Then the three values of z are

$$z_1 = \alpha_1 - \frac{H}{\alpha_1}; \quad z_2 = \alpha_2 - \frac{H}{\alpha_2}; \quad z_3 = \alpha_3 - \frac{H}{\alpha_3}.$$

The corresponding values of x are then found from

$$x = y + h = y - \frac{a_1}{a_0} = \frac{z}{a_0} - \frac{a_1}{a_0} = \frac{z - a_1}{a_0}.$$

Nature of the Roots. — The following criteria serve to determine the nature of the roots:

(a) $G^2 + 4H^3 < 0$, three real distinct roots;
(b) $G^2 + 4H^3 = 0$, three real roots, two being equal;
(c) $G^2 + 4H^3 > 0$, one real root, two imaginary roots.

By direct calculation, for which we shall not take space, we find

$$(z_1 - z_2)(z_2 - z_3)(z_3 - z_1) = \sqrt{-27(G^2 + 4H^3)},$$

or,

$$(z_1 - z_2)^2 (z_2 - z_3)^2 (z_3 - z_1)^2 = -27(G^2 + 4H^3).$$

When the roots are all real, their differences are real, hence the left member of the last equation is positive; therefore $G^2 + 4H^3$ must be negative. When two roots are equal, their difference is zero; hence $G^2 + 4H^3 = 0$. When two roots are imaginary, they must be conjugate imaginaries; suppose them to be

$$z_1 = a + ib \quad \text{and} \quad z_2 = a - ib.$$

Let the third root be $z_3 = c$, where c is real [(1), **(194)**]. Then we show directly that $(z_1 - z_2)^2$ is negative, and that $(z_2 - z_3)^2 (z_3 - z_1)^2$ is positive, hence the left member of the above equation is negative; therefore $G^2 + 4H^3$ must be positive.

The quantity $G^2 + 4H^3$ is called the *discriminant* of the cubic

$$z^3 + 3Hz + G = 0.$$

When all the roots are real, i.e., $G^2 + 4H^3 < 0$, r and s are conjugate complex quantities; let them be

$$r = A + iB; \quad s = A - iB.$$

In this case $\sqrt[3]{r}$ and $\sqrt[3]{s}$ cannot be evaluated algebraically. The roots may then be obtained in trigonometric form. Let

$$A = u\cos v; \quad B = u\sin v.$$

Then

$$r = u(\cos v + i\sin v); \quad s = u(\cos v - i\sin v).$$

Hence

$$\sqrt[3]{r} = \sqrt[3]{u}\left(\cos\frac{v + 2k\pi}{3} + i\sin\frac{v + 2k\pi}{3}\right),$$

$$\sqrt[3]{s} = \sqrt[3]{u}\left(\cos\frac{v + 2k\pi}{3} - i\sin\frac{v + 2k\pi}{3}\right); \quad k = 0, 1, 2.$$

Here $\sqrt[3]{u}$ denotes the real cube root of u.

We now find

$$z = \sqrt[3]{r} + \sqrt[3]{s} = 2\sqrt[3]{u}\cos\frac{v + 2k\pi}{3}; \quad k = 0, 1, 2.$$

197. Ferrari's Solution of the Quartic Equation. — Write the given quartic equation in the form

$$(1) \qquad x^4 + 2ax^3 + bx^2 + 2cx + d = 0.$$

Add to both members $(px + q)^2$:

$$(2) \quad x^4 + 2ax^3 + (b + p^2)x^2 + 2(c + pq)x + (d + q^2) = (px + q)^2.$$

The left member will become a perfect trinomial square of the form

$$(x^2 + ax + k)^2$$

by putting

$$(3) \quad p^2 = a^2 - b + 2k; \quad q^2 = -d + k^2; \quad pq = -c + ak.$$

Then equation (2) becomes

$$(x^2 + ax + k)^2 = (px + q)^2,$$

or,

$$(4) \qquad x^2 + ax + k = \pm(px + q).$$

Taking each sign in turn we have two quadratic equations in x, which give the four roots of (1).

To obtain the values of p, q, and k in (4) we must solve equations (3) for these quantities in terms of the coefficients. On equating the values of p^2q^2 from the product of the first two of equations (3) and the square of the third equation we find a cubic to determine k:

$$(5) \qquad 2k^3 - bk^2 + 2(ac - d)k + (bd - a^2d - c^2) = 0.$$

This is called the *reducing cubic*, and is to be solved for a real value of k. Then p and q are obtained from (3).

Example. $x^4 + 4x^3 - 3x^2 - 16x + 5 = 0.$

Here $a = 2,\ b = -3,\ c = -8,\ d = 5.$

Then (5) is $2k^3 + 3k^2 - 42k - 99 = 0.$

A real root is $k = -3.$

Then from (3), $p = 1,\ q = 2;$ or, $p = -1,\ q = -2.$

With either set of values of p and q (4) becomes

$$(x^2 + 2x - 3) = \pm(x + 2).$$

Hence

$$x^2 + x - 5 = 0, \quad \text{or,} \quad x^2 + 3x - 1 = 0.$$

Therefore

$$x = \frac{-1 \pm \sqrt{21}}{2}, \quad \text{or,} \quad \frac{-3 \pm \sqrt{13}}{2}.$$

Exercises. Solve the following equations:

1. $x^3 - 3x^2 + 4 = 0.$
2. $x^3 - 3x - 2 = 0.$
3. $4x^3 - 3x - 1 = 0.$
4. $x^3 - 24x - 48 = 0.$
5. $x^3 - 7x^2 + 4x + 24 = 0$
6. $x^3 - 3x^2 - 6x + 1 = 0.$
7. $x^3 - 7x - 6 = 0.$
8. $x^3 - x^2 + x - 1 = 0.$
9. $x^4 + 2x^3 + 2x^2 - 2x - 3 = 0.$
10. $x^4 + 6x^3 + 3x^2 - 2x - 3 = 0.$
11. $x^4 - 4x^3 - 9x^2 + 2x + 3 = 0.$
12. $x^4 + 4x^3 - 16x + 11 = 0.$
13. $x^4 + 4x^3 - 16x - 16 = 0.$
14. $x^4 - 3x^3 - 7x^2 + 15x + 18 = 0.$
15. $x^4 - 4x^3 - 8x + 32 = 0.$
16. $x^4 + x^3 - 2x^2 - 3x - 3 = 0.$

ANSWERS

(For most of the odd-numbered exercises.)

Article 10 — Page 8

1. $\frac{5}{4}a - \frac{3}{4}$. **3.** $\frac{31}{20}a^2 - 3.94\,ab - \frac{289}{8}\,ac$. **5.** $47\,x - 2\,y$. **7.** $a^2b^2c - 4\,a^2c^3 + \frac{4}{15}\,a^3cd^2 - 2\,a^2c$. **9.** $.8\,a^3bc^2d^4 - 1.2\,ac^3d^7 + .2\,a^2c^5d^4 - 2\,ac^2d^4$. **11.** $x^6 - 5\,x^4 - x^3 + 6\,x^2 + x - 2$. **13.** $x^7 - 9\,x^5y^2 + 5\,x^4y^3 + 13\,x^3y^4 - 5\,x^2y^5 - 4\,xy^6 + y^7$. **15.** $a^3/4 + a/3$. **17.** $x^4 - (a^2 + b^2)\,x^2 + a^2b^2$. **19.** $5\,a^2 + 7\,a - 6$. **21.** $-3\,m^2p/2$. **23.** $\frac{-30\,xy^2}{31}$. **25.** $\frac{7\,(x^4 - 2\,x^2 + 1)}{100\,x^2}$. **27.** $\frac{-10}{y} + \frac{25\,y^2}{24\,x^2} + \frac{5\,y^3}{12\,x^3} - \frac{5}{2\,x^3}$. **29.** $3\,a^2x - 4\,ax^2 + x^3$. **31.** $x^2 + 15\,xy + 27\,y^2 + 162\,y^3/(2\,x - 9\,y)$. **33.** $\frac{3}{8}\,a^3 + \frac{2}{3}\,a^2b + \frac{1}{2}\,ab^2$. **35.** $x^3 - \frac{8}{5}\,x^2 - 2\,x + \frac{1}{3}$. **37.** $2\,x^2 - xy - 3\,y^2$. **39.** $(a - b)^3$. **41.** $x^2/9 + 3\,y$. **43.** $ab - c$. **45.** $a^2 + b^2 + c^2 + d^2 - 2\,(ab - ac + ad - bd + bc + cd)$.

Article 12 — Page 10

1. $\frac{x}{2\,a}\left(x - \frac{y}{4\,a^2} - \frac{1}{2\,a}\right)$. **3.** $(3\,x + 1)\,(x + 1)$. **5.** $(3\,x + y)\,(2\,x - 7\,y)$. **7.** $x\,(2\,x + 3\,y)\,(4\,x^2 - 6\,xy + 9\,y^2)$. **9.** $(x + 2)\,(x - 2)\,(x + 3)\,(x - 3)$. **11.** $(x - 10)\,(x + 11)$. **13.** $(x + a^2)\,(x + 9\,a^2)$. **15.** $(xy + 5\,z)\,(xy - 2\,z)$. **17.** $(x - 1)\,(x + 6)\,(x - 6)$. **19.** $(x^2 + 1)\,(x^4 - x^2 + 1)$. **21.** $-3\,xy\,(x + y)$. **23.** $(ac + d)\,(ac - b)$. **25.** $xy\,(x - y)\,(x + y)^2$. **27.** $(x^2y + 5)\,(x^2y - 22)$. **29.** $(x + 2)\,(x^2 + 7\,x + 2)$.

Article 15 — Page 12

1. $3\,(x + 1)$. **3.** $4\,(x^2 + y^2)$. **5.** $ax\,(a - x)^2$. **7.** $3\,a\,(2\,a + 3\,b - 4\,c)$. **9.** $(2\,x - 3\,y)$. **11.** $(3\,x - 2\,a)$. **13.** $(x^2 + 7)$. **15.** $(5\,x^2 - 1)$. **17.** $(x + y)$. **19.** $(a^2 - ab + b^2)$. **21.** $24\,a^2bx^2y^3$. **23.** $(a + b)\,(a - b)^2$. **25.** $(x - 4)\,(x + 1)\,(x + 3)$. **27.** $(3\,x - 2)\,(2\,x + 3)\,(2\,x - 3)$. **29.** $(3\,x - 2\,a)\,(4\,x - 3\,a)\,(3\,x + 4\,a)$. **31.** $(m + n)\,(m - n)\,(m + 2\,n)\,(m - 2\,n)$. **33.** $(x + 1)\,(x + 2)\,(x + 3)$. **35.** $(x - 1)\,(x + 1)\,(x + 2)\,(x + 3)$. **37.** $(x - 1)\,(x + 1)\,(x^2 + 1)\,(x^2 - x + 1)$. **39.** $(a - b)\,(a + b)\,(a^2 + ab + b^2)\,(a^2 - ab + b^2)\,(a^6 + a^3b^3 + b^6)$.

Article 19 — Page 15

1. $4\,ax$. **3.** $\frac{a^2 - b^2}{x + y}$. **5.** $\frac{x^8 - x^6y^2 + x^4y^4 - x^2y^6 + y^8}{x^4y^4}$. **7.** 1. **9.** $\frac{x^2 - x + 1}{x^2 + 1}$. **11.** $\frac{x^4 - x^2y^2 + y^4}{3\,(x - y)}$. **13.** $\frac{x\,(a - 2\,x)}{a^2}$. **15.** $-\frac{v^2}{u^2}$. **17.** $x^3 - y^3$. **19.** $\frac{4}{3\,y + 3}$. **21.** 0.

23. $\frac{2}{(x-1)(x-2)(x-3)}$. **25.** $\frac{-(3x^4-2x^3+3x-4)}{2(x+1)(x^2+1)(x^2-x+1)}$. **27.** $\frac{x^3}{a^3}-\frac{x^2}{a^2}+\frac{a^2}{x^2}-\frac{a^3}{x^3}$. **29.** $\frac{7x}{11y}\left(1-\frac{343x^3}{1331y^3}\right)$. **31.** $\frac{(5a^3-9c^3)(45c^3-49b^3)(9c^3-5a^3)}{14{,}175a^3c^6}$.

Article 25 — Page 19

1. 108. **3.** 1152. **5.** a^{q+p}. **7.** 3^4 or 81. **9.** $8a^3$. **11.** b^{-5}. **13.** $6a^5b^7c^9$. **15.** $3p^3q^2r^{-1}$. **17.** $x^{-4}-y^{-4}$. **19.** $64a^{12}$.

Article 29 — Page 21

1. 8. **3.** $4a$. **5.** $2x^{\frac{2}{3}}$. **7.** $a-b$. **9.** $\sqrt{x}+\sqrt{y}$. **11.** $a^{\frac{1}{3}}-b^{\frac{1}{3}}$. **13.** $x^{\frac{1}{2}}$. **15.** $a^{\frac{1}{6}}$. **17.** $a^{\frac{1}{2}}b^{\frac{1}{3}}$.

Article 39 — Page 27

1. $\frac{3}{5}ab^2c$. **3.** $\frac{a^3b^3}{(b-a)^3}$. **5.** $\frac{y^2}{bx^2}$. **7.** $\frac{1}{x^3a^4b^6}$. **9.** $\sqrt[6]{27}, \sqrt[6]{16}$. **11.** $\sqrt[15]{32}, \sqrt[15]{27}$. **13.** $\sqrt[12]{729}, \sqrt[12]{16}, \sqrt[12]{125}$. **15.** $\sqrt[12]{\frac{1}{16}}, \sqrt[12]{\frac{125}{27}}, \sqrt[12]{\frac{729}{15625}}$. **17.** $\sqrt[14]{\frac{25}{49}}, \sqrt[14]{\frac{128}{2187}}, \sqrt[14]{\frac{2}{11}}$. **19.** $\sqrt[12]{a^6}, \sqrt[12]{b^3}, \sqrt[12]{c^2}$. **21.** $\sqrt[36]{a^{24}}, \sqrt[36]{a^{27}}, \sqrt[36]{a^{20}}$. **23.** $\sqrt[30]{\frac{m^{10}}{n^{15}}}, \sqrt[30]{\frac{1}{y^{24}}}, \sqrt[30]{\frac{n^{20}}{y^{40}}}$. **25.** $\sqrt[mnp]{x^{-2np}}, \sqrt[mnp]{x^{mp}}, \sqrt[mnp]{x^{-3mn}}$. **27.** $-\sqrt[3]{4}$. **29.** $7\sqrt{2}$. **31.** $\frac{11}{2}\sqrt[3]{5}$. **33.** $(3+b-a^2)\sqrt{a}$. **35.** $a\sqrt{x}(1-ax+ax^2)$. **37.** $1+\sqrt{5}$. **39.** $2+\sqrt{3}$. **41.** $\frac{1}{2}(\sqrt{14}+\sqrt{6})$. **43.** $\sqrt{5}-\sqrt{2}$. **45.** $\sqrt{10}+\sqrt{3}$. **47.** 17. **49.** $6\sqrt{2}+8\sqrt{3}-6\sqrt{6}-9$. **51.** $6+\sqrt[3]{18}-12\sqrt[3]{12}$. **53.** $\sqrt{m^2-n}$. **55.** $a^{\frac{1}{4}}$. **57.** $\sqrt[15]{x^{32}y^{37}}$. **59.** $\sqrt{7}$. **61.** 2. **63.** 3. **65.** $\frac{1}{2}\sqrt[4]{8}$. **67.** $\sqrt[35]{\frac{a^{18}}{32}}$. **69.** $\frac{1}{3z^2\sqrt{2}}$. **71.** $m^{\frac{2}{3}}$. **73.** $a^{\frac{25}{4}}$. **75.** $-a^6$. **77.** $a^{\frac{25}{6}}$. **79.** $(x+y)^{\frac{2}{3}}$. **81.** $\frac{1}{2}\sqrt{2}$. **83.** $\sqrt[3]{a}$. **85.** $\frac{a(2+\sqrt{a})}{4-a}$. **87.** $\frac{4a^3-12a^{\frac{3}{2}}+9}{9-4a^3}$. **89.** $\frac{1}{5}(3\sqrt{14}-11)$. **91.** $\frac{a\sqrt{b}+c\sqrt{d}}{a^2b-c^2d}$. **93.** a^2. **95.** $2a^{-2}-7a^{-\frac{5}{3}}+6a^{-\frac{4}{3}}+5a^{-1}-11a^{-\frac{2}{3}}+2a^{-\frac{1}{3}}+a^{-\frac{1}{2}}-6$. **97.** $2a^{\frac{5}{x}}-5a^{\frac{4}{x}}-2a^{\frac{3}{x}}-a^{\frac{2}{x}}-6a^{\frac{1}{x}}$. **99.** $4a^{-4}b^{-\frac{4}{3}}-12a^{-\frac{13}{6}}b^{-\frac{11}{12}}+9a^{-\frac{1}{3}}b^{-\frac{1}{2}}$. **101.** $x^{\frac{3}{2}}-3x^2y^{\frac{1}{3}}+3x^{\frac{1}{2}}y^{\frac{2}{3}}-y$. **103.** $m^{-8}+4m^{-\frac{13}{2}}+6m^{-5}+4m^{-\frac{7}{2}}+m^{-2}$. **105.** $a^{\frac{4}{3}}+a^{\frac{3}{2}}+a^{\frac{8}{5}}+2(a^{\frac{17}{12}}-a^{\frac{22}{15}}-a^{\frac{31}{20}})$. **107.** $a^{\frac{1}{2}}+4b^{\frac{2}{3}}+9c+4d^{\frac{2}{3}}+2(-2a^{\frac{1}{4}}b^{\frac{1}{3}}+3a^{\frac{1}{4}}c^{\frac{1}{2}}-2a^{\frac{1}{4}}d^{\frac{1}{3}}-6b^{\frac{1}{3}}c^{\frac{1}{2}}+4b^{\frac{1}{3}}d^{\frac{1}{3}}-6c^{\frac{1}{2}}d^{\frac{1}{3}})$. **109.** $(u^4v-3u^3v^2+3u^2v^3-2uv^4)^2$ or $u^2v^2(u^3-3u^2v+3uv^2-2v^3)^2$. **111.** $x^{\frac{2}{3}}-x^{\frac{1}{3}}+1$. **113.** $a^{\frac{1}{6}}+a^{\frac{1}{12}}b^{\frac{1}{12}}+b^{\frac{1}{6}}$. **115.** $a-\sqrt{a}$. **117.** $3\sqrt{-1}, 5\sqrt{-1}, 9\sqrt{-1}$. **119.** $4i\sqrt{5}$. **121.** $8i\sqrt{3}$. **123.** $9i$. **125.** ni. **127.** $41i-23$. **129.** $4i\sqrt{6}-2$. **131.** -1. **133.** i. **135.** $\frac{a^2-x}{a^2+x}+i\frac{2a\sqrt{x}}{a^2+x}$. **137.** $-\frac{1}{2}$. **139.** $2x=3$. **141.** $ax+b=c^4$. **143.** $4x=5$. **145.** $x=5$. **147.** $x=10$. **149.** $x=4$.

Article 43 — Page 34

1. 0. **3.** $-3, -4, -6, -7$. **5.** $-3, -4$. **7.** $2, a$. **9.** $7, 0.3$. **13.** $(p+q)\left(1+\frac{1}{pq}\right)$. **17.** $\sqrt[3]{3}$; $\sqrt[m]{m}$.

Article 46 — Page 42

1. 2.9196, 0.9196, 9.9196 − 10, 8.9196 − 10. **3.** 410.5; 29.65; 1.664. **5.** 3.667. **7.** 1.655; 11.695. **9.** 52.22; 29.34. **11.** 0.1829. **15.** $\log\frac{ac}{bd}$. **17.** $\frac{1}{12}\log\frac{u^6v^4}{w^3}$. **19.** $\log\frac{a^3\sqrt[6]{ax+b}}{x+y}$. **21.** 2.424 in. **23.** 10,470 lb. **25.** 2.423 in.

Article 48 — Page 44

1. \$2970. **3.** \$2208. **5.** \$8,622,000,000.

Article 53 — Page 48

1. $x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$. **3.** $a^{-4} + 4a^{-3}b^{-2} + 6a^{-2}b^{-4} + 4a^{-1}b^{-6} + b^{-8}$. **5.** $x^4 + 6x^{\frac{43}{12}} + 15x^{\frac{19}{6}} + 20x^{\frac{11}{4}} + 15x^{\frac{7}{3}} + 6x^{\frac{23}{12}} + x^{\frac{3}{2}}$. **7.** $a^8x^8 + 8a^7x^7by + 28a^6x^6b^2y^2 + 56a^5x^5b^3y^3 + 70a^4x^4b^4y^4 + 56a^3x^3b^5y^5 + 28a^2x^2b^6y^6 + 8axb^7y^7 + b^8y^8$. **9.** $8x^3 - 24x^2\sqrt[6]{72x^5} + 60x^2\sqrt[3]{9x^2} - 120x^2\sqrt{2x} + 90x^2\sqrt[3]{3x} - 18x^2\sqrt[6]{648x} + 9x^2$. **11.** $25m^{-4} + 20m^{-3}n^{-2} + 6m^{-2}n^{-4} + \frac{4}{5}m^{-1}n^{-6} + \frac{1}{25}n^{-8}$. **13.** $a^{5x} + 5a^{4x}b^y + 10a^{3x}b^{2y} + 10a^{2x}b^{3y} + 5a^xb^{4y} + b^{5y}$. **15.** $x^{6n^2} - 6x^{5n^2}y^{m^2} + 15x^{4n^2}y^{2m^2} - 20x^{3n^2}y^{3m^2} + 15x^{2n^2}y^{4m^2} - 6x^{n^2}y^{5m^2} + y^{6m^2}$. **17.** $x^{\frac{8}{a}} + 8x^{\frac{7}{a}}y^{\frac{1}{b}} + 28x^{\frac{6}{a}}y^{\frac{2}{b}} + 56x^{\frac{5}{a}}y^{\frac{3}{b}} + 70x^{\frac{4}{a}}y^{\frac{4}{b}} + 56x^{\frac{3}{a}}y^{\frac{5}{b}} + 28x^{\frac{2}{a}}y^{\frac{6}{b}} + 8x^{\frac{1}{a}}y^{\frac{7}{b}} + y^{\frac{8}{b}}$. **21.** $(x+y)^3 - 3(x+y)^2z + 3(x+y)z^2 - z^3$. **23.** $(12\sqrt{2} - 15) + 4i(5\sqrt{2} - 6)$. **27.** $2880c^3d\sqrt{6cd}$. **29.** $\left(\frac{5005}{27}\right)y^{15}$.

Article 54 — Page 49

1. 0.975. **3.** 88.444. **5.** 4083.727. **7.** 0.9996500.

Article 58 — Page 52

1. $x = \frac{5a-3b}{2}$. **3.** $x = \frac{c}{b}$. **5.** $\frac{m-p}{q-n}$. **7.** ∞. **9.** $\frac{b}{b-a-1}$. **11.** $\frac{mn}{m+n-a}$; ∞. **13.** 1. **15.** $\frac{1}{4}$.

PROBLEMS.

1. 6. **3.** $\frac{ma}{m-1}$. **5.** $\frac{a}{n-m}$. **7.** 3 to 1. **9.** $1\frac{7}{8}$ days. **11.** $\frac{abc}{ab+ac+bc}$ days. **13.** $5\frac{5}{11}$ min. past 10; $21\frac{9}{11}$ min. past 10. **15.** $1\frac{1}{2}$ hrs.

Article 70 — Page 61

1. 3, 5. **3.** 5, −4. **5.** 9, −8. **7.** 2, 3. **9.** Inconsistent. **11.** 0, −4. **13.** 1, −3. **15.** Dependent. **17.** 6, −12. **19.** 4, 5.

Article 73 — Page 63

1. 6, 6. **3.** 6, 12. **5.** 9, 7. **7.** 4, $\frac{3}{2}$. **9.** $x = 990.07 +$; $y = 1000.24 +$. **11.** Not independent. **13.** 5, −6. **15.** $\frac{1}{2}$, $\frac{3}{4}$. **17.** 2, 7. **19.** 7, $\frac{2}{7}$. **21.** $\frac{5a-5b}{2}, \frac{a+b}{4}$. **23.** $\frac{abcg}{bg-af}, \frac{abcf}{bg-af}$. **25.** $x = \frac{nqrt+npsv}{mqr+ps}$; $y = \frac{msqv-qst}{ps+mqr}$. **27.** $\frac{m^2q}{mq-np}, \frac{nmq}{mq-np}$. **29.** $\frac{117}{4}$, $\frac{13}{4}$. **31.** No solution. **33.** 20, 17, 5. **35.** 3, 2, 1. **37.** 3, 4, 5. **39.** $\frac{1}{4}$, $\frac{1}{2}$, ∞. **41.** 1, 2, 3, 4. **43.** 1, .8, .2, .6. **45.** $16\frac{1}{12}$ hrs., $7\frac{11}{12}$ hrs. **47.** $4000; $4\frac{1}{2}$%. **49.** 30, 10. **51.** 89, 35. **53.** 13, 17, 20. **55.** 1, $1\frac{3}{4}$, $1\frac{1}{2}$. **57.** 2, 3, 6 hrs. **59.** $9150, $8600, $7550.

Article 76 — Page 69

1. 3, −7. **3.** 1, −7. **5.** 8, −3. **7.** $\frac{14}{3}$, −3. **9.** −5, $-\frac{3}{2}$. **11.** $\frac{3}{2}$, $-\frac{1}{10}$. **13.** $-\frac{5}{2}$, −5. **15.** −3, $\frac{33}{7}$. **17.** $\frac{1}{2}b\left(-1 \pm \sqrt{-7}\right)$. **19.** $\frac{1}{2}\left(c+2 \pm \sqrt{c^2+12c+4}\right)$.

Article 85 — Page 75

1. 7. **3.** −1 or 2. **5.** 2. **7.** 14. **9.** 3. **11.** 4 or $\frac{13}{2}$. **13.** $\frac{5}{3}$. **15.** $\frac{3}{2}$ or $-\frac{1}{3}$. **17.** $\frac{9}{2}$. **19.** $\frac{1225}{144}$. **21.** ±3. **23.** 3 or $-\frac{5}{4}$. **25.** $\frac{3}{8}$. **27.** $\pm mpi$. **29.** $\pm b\sqrt{a^2-b^2}$. **31.** 0, $\pm 2ai$. **33.** 0. **35.** $m\left[1 \pm 2\sqrt{a}/(a+1)\right]$. **37.** $b \pm \sqrt{b^2-ab}$. **39.** $a\sqrt{a/(a-1)}$. **41.** ±8, $\pm 27i$. **43.** $(a-b)/2 \pm [(a+b)\sqrt{c^2-4}]/2c$. **45.** $-4\sqrt[5]{4}$ or 8. **47.** 4 or −9. **49.** 27 or 64. **51.** $[(1 \pm \sqrt{5})^m - 2^m] \div [(1 \pm \sqrt{5})^m + 2^m]$. **53.** 19, 21, 23; or −19, −21, −23. **55.** 32 × 50. **57.** $\frac{1}{2}\sqrt{ab-A}$. **59.** $\frac{1}{2}\left(b-a \pm \sqrt{a^2+b^2-6ab}\right)$. **61.** $\frac{5}{2}$. **63.** $10/(5n-1)$, $n < \frac{1}{5}$. **67.** $\frac{5}{4}$ or 5 sec. No; t becomes imaginary. Max. $h = 156\frac{1}{4}$ ft., in $3\frac{1}{8}$ sec. **69.** $x > -1$ and < -9; −1 and −9; $x < -1$ and > -9.

Article 91 — Page 81

1. $\pm\frac{1}{2}\sqrt{2}$; $\pm\frac{1}{2}\sqrt{2}$. **3.** $\frac{1}{2}\sqrt{2}$; $-\frac{1}{2}\sqrt{2}$. **5.** $\frac{-6 \pm \sqrt{11}}{5}$; $\frac{3 \pm 2\sqrt{11}}{5}$. **7.** 0 or $\frac{72}{25}$; 3 or $\frac{21}{25}$. **9.** $\frac{6 \pm \sqrt{6}}{20}$; $\frac{-2 \pm 3\sqrt{6}}{20}$. **11.** $m = \pm 2$.

Article 93 — Page 82

1. $\pm\frac{6}{\sqrt{13}}$; $\mp\frac{6}{\sqrt{13}}$. **3.** $\frac{9}{\sqrt{13}}$; $\frac{4}{\sqrt{13}}$. **5.** $0, 2; 1, 0$. **7.** $\frac{192\pm3\sqrt{-1559}}{145}$; $\frac{-162\pm2\sqrt{-1559}}{145}$. **9.** $\frac{4\pm\sqrt{-23}}{13}$; $\frac{-9\pm\sqrt{-23}}{13}$. **11.** $\pm\frac{1}{3}\sqrt{5}$.

Article 95 — Page 83

1. $0, 1; 0, 1$. **3.** $\frac{65\pm\sqrt{129}}{32}$; $\frac{1\pm\sqrt{129}}{16}$. **5.** $\frac{7\pm4\sqrt{-2}}{9}$; $\frac{2\mp4\sqrt{-2}}{3}$. **7.** $-1\pm\sqrt{5}$; $\frac{3\mp\sqrt{5}}{2}$. **9.** $-4\pm2\sqrt{3}$; $-7\pm4\sqrt{3}$. **11.** 1.

Article 97 — Page 84

1. $1, -\frac{5}{4}; 0, -\frac{3}{4}$. **3.** $\frac{1}{2}\sqrt{5}$; $\frac{1}{2}$. **5.** $\frac{18\pm3\sqrt{-34}}{35}$; $\frac{-2\mp12\sqrt{-34}}{35}$. **7.** $\frac{54\pm\sqrt{66}}{25}$, $\frac{-12\mp3\sqrt{66}}{25}$. **9.** $\pm\frac{12}{7}\sqrt{-7}$, $\pm\frac{6}{7}\sqrt{-7}$. **11.** $\pm\sqrt{5}$.

Article 103 — Page 90

1. $x = \pm\frac{1}{2}\sqrt{2}$, $\pm\frac{2}{5}\sqrt{5}$, $y = \pm\frac{1}{2}\sqrt{2}$, $\mp\frac{1}{5}\sqrt{5}$. **3.** $x = 0, 3, \pm\frac{6}{13}\sqrt{13}$; $y = 2, 0, \mp\frac{6}{13}\sqrt{13}$. **5.** $x = 0, 9, \frac{64}{9}$; $y = 0, -6, \frac{16}{3}$.

Article 104 — Page 92

1. $\pm\frac{1}{2}\sqrt{29\pm\sqrt{41}}$, $\pm\frac{1}{2}\sqrt{7\mp\sqrt{41}}$. **3.** $0, \pm\frac{2}{5}\sqrt{5}; 2i, \mp\frac{4}{5}\sqrt{5}$. **5.** $\pm\sqrt{3}$, $\pm\frac{5}{19}\sqrt{57}$, 0, $\mp\frac{2}{19}\sqrt{57}$.

Article 105 — Page 93

1. $\pm\sqrt{3}$; ±1. **3.** $\pm\frac{\sqrt{5}}{3}$; $\pm\frac{4}{3}$. **5.** $\pm\frac{\sqrt{65}}{3}$; $\pm\frac{4i}{3}$.

Article 110 — Page 97

(*Note.* Form all possible combinations of the unknowns where not otherwise indicated.)

1. $(x, y) = (\pm24, \pm16)$. **3.** $(x, y) = (\pm\frac{5}{2}, \pm4)$. **5.** $(x, y) = \left(1\pm\sqrt{5}, \frac{1}{2}[1\mp\sqrt{5}]\right)$; both upper or both lower signs only. **7.** $(x, y) = (\frac{2}{3}, -\frac{1}{2})$; $(\frac{1}{2}, -\frac{2}{3})$. **9.** $(x, y) = (7, 8)$; $(\frac{499}{135}, 9\frac{199}{665})$. **11.** $(13, 14)$. **13.** $(x, y) = (5, -1)$; $(-5, 1)$. **15.** $(x, y) = (7, -15)$; $(-\frac{6}{5}, 87\frac{1}{2})$. **17.** $(x, y) = (8, 7)$; $(77\frac{1}{11}, -44\frac{9}{11})$. **19.** $(x, y) = (2, 4)$; $(\frac{5}{2}, 3)$. **21.** $(x, y) = (14\frac{51}{71}, 31\frac{15}{71})$; $(5, 4)$. **23.** $(x, y) = (8, -9)$; $(9, -8)$. **25.** $(x, y) = (\frac{3}{2}, 2)$; $(2, \frac{3}{2})$. **27.** $(x, y) = \left(\pm1\pm\sqrt{5}, \mp1\pm\sqrt{5}\right)$; use like combinations of signs. **29.** $(x, y) =$

$(3 \pm \sqrt{6}, 3 \mp \sqrt{6})$; $[\frac{1}{2}(-1 \pm \sqrt{-11}), \frac{1}{2}(-1 \mp \sqrt{-11})]$; upper or lower signs only. **31.** $(x, y) = (-2, 0)$; $(\infty, -\infty)$. **33.** $(u, v) = (14, -4)$; $(4, -14)$. **35.** $(r, s) = (0, -10)$; $(5, 0)$. **37.** $(u, v) = (5, 11)$; $(5, -12)$; $(-6, 11)$; $(-6, -12)$. **39.** $(x, y) = (2, -3)$; $(3, -2)$; $(-3 \pm \sqrt{3}, 3 \pm \sqrt{3})$; upper or lower signs only. **41.** $(x, y) = (12, 3)$; $(3, 12)$; $(-8 \pm 2\sqrt{7}, -8 \mp 2\sqrt{7})$; upper or lower signs only. **43.** $(u, v) = (18, 4)$; $(\frac{8}{3}, 27)$; $[\frac{1}{2}(-21 \pm \sqrt{249}), \frac{1}{2}(-21 \mp \sqrt{249})]$; upper or lower signs only. **45.** $(p, q) = (5, 3)$; $(3, 5)$. **47.** $(p, q) = (4, 3)$; $(3, 4)$; $[\frac{1}{2}(7 \pm i\sqrt{295}), \frac{1}{2}(7 \mp i\sqrt{295})]$; upper or lower signs only. **49.** $(u, v) = (4, 7)$; $(7, 4)$; $[\frac{1}{2}(11 \pm i\sqrt{735}), \frac{1}{2}(11 \mp i\sqrt{735})]$; upper or lower signs only. **51.** $(u, v) = (\pm 4, \pm 1)$; $(\pm\frac{2}{43}\sqrt{129}, \mp\frac{11}{258}\sqrt{129})$; upper or lower signs only. **53.** $(x, y) = (14, -4)$; $(-4, 14)$; $(14w, -4w)$; $(-4w, 14w)$; $(14w^2, -4w^2)$; $(-4w^2, 14w^2)$; here $w = \frac{1}{2}(-1 + i\sqrt{3})$, so that $w^3 = 1$. **55.** $(m, n) = (11, -9)$; $(-9, 11)$; $(11w, -9w)$; $(-9w, 11w)$; $(11w^2, -9w^2)$; $(-9w^2, 11w^2)$; $w = \frac{1}{2}(-1 \pm i\sqrt{3})$. **57.** $x = 243, 32$; $y = 64, 729$; two answers. **59.** $x = 3, -1, +1, -3$; $y = 1, -3, 3, -1$. **61.** $\pm\sqrt{1 \pm \frac{1}{2}\sqrt{3}}$; $\pm\sqrt{1 \mp \frac{1}{2}\sqrt{3}}$. Use both upper or both lower signs under radicals; outside of radicals use all combinations. **63.** $\frac{\pm\sqrt{m^2 + 4n^2} + m}{2}$; $\frac{\pm\sqrt{m^2 + 4n^2} - m}{2}$; two solutions. **65.** $\frac{b^2 \pm a\sqrt{2b^2 - a^2}}{2}$; $\frac{b^2 \mp a\sqrt{2b^2 - a^2}}{2}$; two solutions. **67.** $\frac{m}{2a}\left(\pm\sqrt{\frac{n + abm}{n - 3abm}} + 1\right)$; $\frac{m}{2b}\left(\pm\sqrt{\frac{n + abm}{n - 3abm}} - 1\right)$; two solutions. **69.** $x = \pm\sqrt{-5} + 1, y = \pm\sqrt{-5} - 1$; $x = \pm\sqrt{-1} + 1$, $y = \pm\sqrt{-1} - 1$; four solutions. Use upper signs or lower signs only. **71.** $(x, y) = (\frac{1}{2}, \frac{1}{3})$; $(\frac{1}{3}, \frac{1}{2})$. **73.** $ab\,(R - b)/(a^2 - b^2)$; $ab\,(R + b)/(a^2 - b^2)$; $R = \pm\sqrt{2a^2 - b^2}$. **75.** $\frac{p}{\sqrt[3]{p^2 + q^2}}$; $\frac{q}{\sqrt[3]{p^2 + q^2}}$. **77.** $5, 3, 4 \pm \sqrt{-33}$; $3, 5, 4, \mp\sqrt{-33}$. **79.** $x = \pm\sqrt{-3}$, $y = \mp\sqrt{-3}$, $z = 2$; two solutions. **81.** $x = 2$ or ∞; $y = -\frac{1}{2}$ or -1; $z = 1$ or 0. **83.** $x = \frac{1}{2q}(pq - r \pm \sqrt{(pq - r)^2 - 4q^3})$; $y = \frac{1}{2q}(pq - r \mp \sqrt{(pq - r)^2 - 4q^3})$; $z = \frac{r}{q}$; two solutions. **85.** $\pm\frac{19}{4}, \mp\frac{11}{4}, \pm\frac{13}{4}$; take all upper or all lower signs. **87.** $x = [a - b - c - 1 \pm \sqrt{(1 + b + c - a)^2 + 4a(1 + b)(1 + c)}] \div 2(1 + b)$. **89.** $x = \frac{1}{3}$ or $\frac{2}{3}$; $y = \pm\frac{1}{4}\sqrt{-\frac{2}{3}}$, or $\pm\frac{3}{4}$; $z = \pm\frac{1}{2}\sqrt{-\frac{5}{3}}$, or $\pm\frac{1}{2}$.

PROBLEMS

1. 9, 12. **3.** 24, 72. **5.** 35, 20 or −21, −36. **7.** $x = 19, -20$; $y = 17, -18$; four answers. **9.** 33, 56. **11.** 19, 23. **13.** 28, 36 ft. sec. **15.** $13\frac{7}{41}$; 45 days. Assume each man's pay proportional to amount of work he does. **17.** 12, 16. **19.** $\frac{9}{17}$. **21.** 3, 5 yds. **23.** $s_1 = 15.4$; 11.7 ft. sec.; $s_2 = 6.8$; 12.2 ft. sec.

Article 112 — Page 101

1. 3. **3.** −3. **5.** 0. **7.** −3. **9.** 1. **11.** −4. **13.** $-\frac{6}{7}$. **15.** $\frac{2}{3}$. **17.** $\frac{\log 3}{\log \frac{25}{8}}$. **19.** $\frac{\log a^2b}{\log a^{-3}b^2}$. **21.** 1, −3. **23.** $-\frac{16}{5}$. **25.** −6.

Article 120 — Page 107

13. 67,300,000 mi.; 487,000,000 mi. **15.** 4.1+ or 24.1+ ft. from smaller light.

Article 124 — Page 109

1. 55; 403. **3.** 14; 200. **5.** 28; 364. **7.** $p - \frac{19}{2}q$; $20p - 95q$. **9.** $l = 150$; $d = 3$. **11.** $a = 9$; $d = 2$. **13.** $a = 18$; $d = 5$. **15.** $a = 17$; $l = 97$. **17.** $a = \frac{1}{3}$; $l = \frac{50}{3}$. **19.** $n = 16$, $l = 69$. **21.** $n = 14$; $a = 12$. **23.** $n = 103$, $a = 1281$. **25.** 8925. **27.** 10 sec. **29.** 29700 ft.

Article 128 — Page 112

1. $l = 256$; $S = 508$. **3.** $l = 4096$; $S = 5461$. **5.** $l = \frac{1}{262144}$; $S = \frac{349525}{262144}$. **7.** $l = a(1+x)^7$; $S = \frac{a(1+x)^8 - a}{x}$. **9.** ±48; 288, ±1728. **11.** ±12, 4, $\pm\frac{4}{3}$, $\frac{4}{9}$, $\pm\frac{4}{27}$. **13.** 12, 3, $\frac{3}{4}$, $\frac{3}{16}$. **15.** $a = 2$, $S = 254$. **17.** $a = 6$, $S = \frac{242}{27}$. **19.** $n = 6$, $S = 126$. **21.** $r = 3$, $n = 7$. **23.** $r = \frac{1}{2}$, $n = 6$. **25.** $n = 9$, $l = 19683$. **27.** $a = 5$, $l = 320$.

Article 130 — Page 113

1. $\frac{22}{3}$. **3.** $\frac{25}{2}$. **5.** $\frac{2}{3}$. **7.** $\frac{1}{3}$. **9.** $\frac{16}{99}$. **11.** $\frac{41}{333}$. **13.** 179.99+ ft.

Article 132 — Page 114

1. $a = 115$ or 1; $d = -10$ or $+2$. **3.** $a = -11$ or $\frac{755}{7}$; $d = 4$ or $-\frac{100}{7}$. **5.** $a \pm d$, $a \pm 3d$; $a = \frac{11}{4}$, $d = \frac{1}{12}\sqrt{605 \pm 4\sqrt{135601}}$. **7.** Middle number $= b$; com. diff. $= \pm\sqrt{\frac{5}{8}b^2 \pm \frac{1}{8}\sqrt{9b^4 + \frac{16a}{b}}}$. **9.** 55°, 60°, 65°. **11.** a, $ar^{\frac{1}{2}}$, ar, $ar^{\frac{3}{2}}$, $\cdots$. **13.** $\pm\frac{5}{2}$, ±10, ±40, ±160. **15.** 10.62 inches. **17.** 0.7 gr. (4-place tables), 0.6702 gr. (7-place tables); 1880 years. **19.** $\$1845 \times 10^{10}$. **21.** $2a$; $4a\sqrt{3}$.

Article 134 — Page 117

1. \$1308. **3.** \$713. **5.** \$6500; first payment now due. **7.** \$742.

Article 144 — Page 124

1. Convergent. **3.** Conv. if $|x| < 1$. Div. if $|x| \geqq 1$. **5.** Divergent. **7.** Divergent. **9.** Convergent for all values of x. **11.** Conv. when $|x| < 1$.

Article 151 — Page 133

1. $\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$. **3.** $\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$. **5.** $n\left(a + \frac{n-1}{2}d\right)$.

Article 153 — Page 135

3. .0314; .0204. **5.** $5^h\,16^m\,05^s.59$; $18°\,48'10''.1$.

Article 154 — Page 136

1. Sixth entry should be .364. **3.** Sixth entry should be 3′ 30″.

Article 155 — Page 139

1. $a_n = a_{n-2} - a_{n-1};\ n > 1$. **3.** $1 + x^2 + x^3 + 2x^4 + \cdots$. **5.** $\frac{3}{2}x - \frac{1}{2}x^2 - \frac{1}{4}x^3 + \frac{1}{2}x^4 + \cdots$. **7.** $\frac{2}{3x} - \frac{17}{9} + \frac{83x}{27} - \frac{383x^2}{81} + \cdots$.

Article 160 — Page 143

1. $\frac{3}{8(3x+1)} - \frac{1}{8(x+3)}$. **3.** $\frac{2}{x} - \frac{2}{x+2} + \frac{1}{x-2}$. **5.** $\frac{2\sqrt{3}+3}{6(x-2-\sqrt{3})} - \frac{2\sqrt{3}-3}{6(x-2+\sqrt{3})}$. **7.** $\frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x^2+1)}$. **9.** $\frac{3}{x} + \frac{5-3x}{x^2+4}$. **11.** $1 - \frac{1}{3x} + \frac{x-9}{3(x^2+3)}$. **13.** $-\frac{2}{x} + \frac{2}{x-2} - \frac{3}{(x-2)^2}$. **15.** $\frac{1}{3(x+1)} - \frac{x-2}{3(x^2-x+1)}$. **17.** $\frac{1}{2(x-1)} + \frac{1}{5(x-2)} + \frac{3}{10(x+3)}$. **19.** $\frac{4}{x-2} - \frac{1}{x-1}$. **21.** $\frac{1}{x+1} - \frac{2}{x+2} + \frac{2}{x-2}$.

Article 162 — Page 146

3. $x = \frac{13}{16},\ y = \frac{1}{8},\ z = \frac{5}{16}$. **5.** $x = \frac{10}{11},\ y = 1,\ z = \frac{24}{11}$. **7.** Not independent.

Article 170 — Page 155

1. 0. **3.** 0. **5.** 8. **7.** 398. **9.** 832. **11.** $a_1b_2c_3d_4$. **33.** $u = \frac{49}{100},\ v = -\frac{7}{5}$, $w = \frac{88}{25}$. **35.** Inconsistent.

Article 171 — Page 158

1. 24. **3.** 240.

Article 172 — Page 159

1. 20. **3.** 120. **7.** 190.

Article 174 — Page 160

1. 1260. **3.** 360. **5.** ${}_6C_4 \times {}_{14}C_7 + {}_6C_5 \times {}_{14}C_6 + {}_6C_6 \times {}_{14}C_5 = 71500$. **7.** 63. **9.** 4; there will be three different throws. **11.** 36; there will be 21 different throws.

Article 176 — Page 162

1. $\frac{5}{16}$. **3.** $\frac{5}{42}$. **5.** $\frac{20}{21}$. **7.** $\frac{1}{216}$. **9.** $\frac{5}{72}$. **11.** $\frac{66}{195755}$.

Article 178 — Page 164

1. $\frac{4}{15}$. **3.** $\frac{4}{45}$; $\frac{1}{9}$. **5.** 6. **7.** $\frac{4}{19}$. **9.** $\frac{1}{2}$.

Article 183 — Page 166

1. $x^3 - 6x^2 + 11x - 6 = 0$. **3.** $x^4 - 2x^3 - 4x^2 + 8x = 0$. **5.** $6x^4 - 5x^3 - 5x^2 + 5x - 1 = 0$.

Article 192 — Page 174

1. $x^3 + 2x^2 - 4x - 8 = 0$. **3.** $x^3 - 12x - 14 = 0$. **5.** $x^2 + 2x + 1 = 0$. **7.** $x^2 - 2x - 2 = 0$. **9.** $x^3 + x^2 - 9 = 0$. **11.** $x^3 - 9x^2 + 24x - 16 = 0$. **13.** $x^4 + 6x^3 + x^2 - 24x + 16 = 0$. **15.** $h = 1$; $x^3 - 3x = 0$. **17.** $h = 1$; $x^3 - 9x - 7 = 0$. **23.** $\pm\sqrt{-1}$, 2. **25.** 2, $\pm 2\sqrt{2}$, $-1 \pm \sqrt{-3}$.

Article 197 — Page 182

1. 2, 2, -1. **3.** 1, $-\frac{1}{2}$, $-\frac{1}{2}$. **5.** 3, $2 \pm 2\sqrt{3}$. **7.** $-1, -2, 3$. **9.** $1, -1, -1 \pm \sqrt{-2}$. **11.** $\frac{1}{2}(-1 \pm \sqrt{5})$, $\frac{1}{2}(5 \pm \sqrt{37})$. **13.** 2, -2, -2, -2. **15.** 4, 2, $-1 \pm \sqrt{-3}$.

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